BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 7, 1956

MATHEMATICS

On the ε-Length

by

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Presented by H. STEINHAUS on March 23, 1956

H. Steinhaus has pointed out in [1]-[7] a paradox resulting from the discontinuity of the functional given by the length of a plane curve. By a curve we shall understand here a homeomorphic image of a segment on a plane. Curves will be denoted by the letters X, Y, Z, and real numbers by small Greek letters. By the distance between two curves, X and Y, we shall understand the greatest distance between a point of one of those curves and the other curve.

(0) $\varrho(X, Y) = \max \varrho(z, Z)$, where Z is one of the sets X or Y, and z is an element of the other set.

Discontinuity of length means that for an arbitrarily small number $\mu > 0$ and an arbitrarily large number ν we can choose two rectifiable curves X and Y, distant less than μ one from another and differing in length by more than ν. It will be observed that the area of the region bounded by a closed curve (on a plane) is a continuous functional, i. e., that the areas bounded by curves which lie close to one another in the sense of distance (0), differ very slightly.

According to some opinions, the difficulties concerning length cannot have any practical consequence since empirical curves are always rectifiable. H. Steinhaus points out the significance of the paradox of length for empirical curves. Thus, for instance, the length of a razor edge is 36 mm., but if we view it through a microscope, then, instead of a straight segment, we shall see a curve full of irregularities and much longer than 36 mm. This length would increase still more if the edge were observed through an electronic ultramicroscope. Repeating our observations with more and more accuracy we should measure each time a different curve lying close to those measured previously. The lengths of those curves would make up an increasing sequence, the differences not always decreasing. The "real" length of the edge is the inaccessible limit of this sequence even if it exists, i. e. if the curve is rectifiable.

The observed length of an empirical curve is not connected with its real length expressed mathematically, and we can measure the length of a curve empirically, disregarding the question of whether is is rectifiable or not. Empirical length is a new concept. It should be defined, accepted as a conventional concept, and applied in practice instead of the ideal mathematical concept of length (I owe this idea to H. Steinhaus). In this paper concepts suitable for this purpose are described.

The length of a plane curve may be defined as a plane measure of the set of points of Crofton's plane [8], the straight lines which correspond to them intersecting the measured curve; each point of Crofton's plane is counted as many times as the straight line corresponding to it intersects the curve. On this principle H. Steinhaus has based a simple instrument for measuring the length of curves. The instrument, called a longimeter, is a sheet of squared transparent paper. The curve to be measured is covered with the sheet and the number of points of intersection of the longimeter lines with the curve is counted. The expected number of intersection points is proportional to the length of the curve. This method does not, however, avoid the paradox of length mentioned above.

In order to get rid of this paradox H. Steinhaus has suggested a new concept — length of order k. It is defined as the number obtained by measuring the length of a curve with a longimeter with the following change of procedure; on each line of the longimeter at most k points of intersection with the curve are counted, further points (those over k) not being counted at all. The lengths of ever higher order form a non-decreasing sequence converging to the real length. The concept of length of order k solves a large number of problems connected with measuring the length of empirical curves; it is particularly useful in comparing the length of curves on a map or on several maps differently generalised. There are, however, problems in natural science for which H. Steinhaus's solution is insufficient. The following three reasons justify this view:

1. H. Steinhaus's method is based on a convention. Namely, we must agree as to the order of the length which is to be measured. k is conventional. Obviously, the larger k, the more accurate the measurement, but it may happen that a naturalist is unable to see the connection between the magnitude of k and the accuracy of the measurement and does not know what order of k he should choose in a given case. In some cases it would be better to state expressly the convention, i. e., characterise the accuracy of measurement in units of length or by a segment.

2. H. Steinhaus substitutes for the concept of length a sequence of new concepts, namely the concepts of length of the first order, of the second order, and so on. Naturalists sometimes do not understand and do not want to understand these new notions. They want to believe in the real length of an empirical curve and by its approximate length to

understand the real length of another curve, approximating the one measured. Naturalists wish to know which curves they can measure accurately and which only approximately, and finally they want to know what the approximating curve looks like.

3. Some empirical curves are so varied that if we place under the microscope a point of intersection of such a curve with a line of the longimeter, we would find that the curve intersects the straight line in the vicinity of the point in question not once but many times. If we wanted to measure such a curve by Steinhaus's method, we would find that every straight line intersecting that curve intersects it twice, and three times and also k times. Consequently, the length of order k would only be the k-multiple of the length of the first order, i. e. it would not give better accuracy.

Now we shall introduce a new concept — that of the length of order ε. This concept will contain a conventional parameter ε — the length of a segment. We shall show how the accuracy of measurement depends on ε and explain in terms of old concepts (such as length and area) the meaning of the new length of order ε. We shall show for what curves the length of order ε is the real length and what is the approximate length of other curves. In another paper [9] we shall describe a simple device, similar to H. Steinhaus's longimeter, by means of which the length of order ε can be measured.

In paper [10] the ε-neighbourhood of a curve X was defined as the set of all points of the plane the distance of which from the curve is not greater than ε.

$$(A) \qquad A_\varepsilon(X) = \prod_x [\varrho(x, X) \leqslant \varepsilon].$$

It follows from the properties (1), (2), (4) and (5) of the work quoted above that the set $A_\varepsilon(X)$, regarded as a function of the arguments ε and X, is an increasing function, continuous with respect to each argument. The area $\alpha_\varepsilon(X)$ of the ε-neighbourhood of the curve X is thus also a continuous increasing function of the arguments ε and X.

H. Minkowski ([11], p. 122-127) defines the length of a curve X by the formula.

$$(M) \qquad L(X) = \lim_{\varepsilon \to 0} \frac{\alpha_\varepsilon(X)}{2\varepsilon}.$$

We shall make use of Minkowski's definition without passing to the limit. It will be observed, however, that the ε-neighbourhood of the curve X consists of a strip covering the curve and two semi-circles, the initial point and the end point of the curve being their respective centres (Fig. 1). The

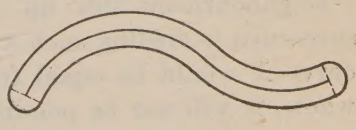

Fig. 1

area of the two semi-circles decreases to 0 with ε^2, and thus plays no part in Minkowski's definition. If, however, we do not pass to the limit, the area of those two semi-circles must be subtracted from the area of the ε-neighbourhood in order to obtain the area of the strip environing the curve X. In this way the following definition of the length of order ε is obtained:

(D) $$L_\varepsilon(X) = \frac{a_\varepsilon(X) - \pi\varepsilon^2}{2\varepsilon}.$$

It follows immediately from the continuity and monotony of the function $a_\varepsilon(X)$ that the function $L_\varepsilon(X)$ is continuous with respect to both ε and X and that it is increasing with respect to the argument X. We find that $L(X)$ is a decreasing function of the argument ε. The proof of this theorem was given by H. Fast [13].

In paper [12] is given the definition of an ε-convex set. Let the curve X be called ε-convex if we can draw, at any point of the curve, tangent circles with diameter ε, on both sides of the curve, having the points of contact as the only points in common with the curve. We find that for any curves X the length of order ε is not greater than the ordinary length, and for 2ε-convex curves the length of order ε is the ordinary length. The proof of these theorems was given by H. Fast [13].

Thus, the length of order ε has the properties which should be required from empirical length. It is a continuous functional and thus removes the paradox of length. For 2ε-convex curves the length of order ε is the ordinary length. For other curves the length of order ε is less than the ordinary length, and increases in a continuous manner as we decrease ε. From the definition (M) it follows that, as ε decreases to 0, the length of order ε tends to the ordinary length, provided the curve is rectifiable. The length of order ε can be regarded as an approximation of the ordinary length and the smaller ε the better the approximation. The ratio ε/L characterises this approximation in a certain sense. E. g., we can conventionally assume that for the measurement of a curve X such an ε should be chosen as to make this ratio suitably small, e. g., 0.1 (H. Steinhaus's idea).

Let us consider any (non-2ε-convex) curve X. We shall try to interpret its length of order ε. It follows from definition (D) that every curve Y with the same ε-neighbourhood as the curve X has the ε-length equal to the ε-length of the curve X. Hence, it follows that the ε-length of the curve X is not greater than the ordinary length of the curve Y, whose ε-neighbourhood fills up the ε-neighbourhood of the curve X. If we succeeded in finding such a 2ε-convex curve Y, then the ε-length of the curve X would be equal to the ordinary length of the curve Y. On the whole, it will not be possible to find such a curve.

The area of the set $A_\varepsilon(X)$ can be measured in a known manner by means of the point planimeter. The expected number of lattice points found in the set $A_\varepsilon(X)$ is proportional to $a_\varepsilon(X)$. But a lattice point will be found in the set $A_\varepsilon(X)$ if, and only if, the circle with radius ε and centre at that lattice point hits upon the curve X. The circle longimeter [9] is based on this principle. The side of the lattice may be chosen for a given ε in such a way as to make the use of the longimeter very simple. For example, for $\varepsilon = 5$ mm. (Fig. 2), we should place the longimeter (on transparent paper) upon the measured curve and count the circles intersected by this curve. This measurement must be repeated 10 times and from the total number of intersected circles we must subtract 8. Thus we obtain (in millimetres) the length of the order $\varepsilon = 5$ mm. of the curve X.

Fig. 2

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POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 7, 1956

MATHEMATICS

Algebraic Characterisation of Abelian Groups which Admits Compact Topologies

by

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Presented by K. KURATOWSKI on March 31, 1956

This note gives a φ algebraic characterisation of Abelian groups, which admits compact topologies.

A certain incomplete characterisation of these groups was given by Kaplansky [2]. An Abelian group D is called divisible if for each integer n and $g \in D$ the equation $nx=g$ is solvable in D. We know [5] that a divisible group D has the form:

$$D=\sum_{\iota} R_{\iota}^{+}+\sum_{p} C_{p}^{\lambda_p}\infty,$$

where R_{ι}^{+} are groups isomorphic to the additive group of rational numbers, and the group $C_{p}^{\lambda_p}\infty$ is the direct sum of λ_p quasicyclic groups $C_p\infty$.

THEOREM 1. *A divisible Abelian group D admits compact topology if, and only if, it is of the form*

$$D=\sum_{\iota\in T} R_{\iota}^{+}+\sum_{p} C_{p}^{\lambda_p}\infty,$$

where: 1. $\overline{\overline{T}}=2^{\aleph}$, $\aleph\geqslant\aleph_0$; 2. λp *is finite, or* $\lambda p=2^{m_p}$, $m_p\geqslant\aleph_0$; 3. *for each* p, $\overline{\overline{T}}\geqslant\lambda p$.

THEOREM 2. *An arbitrary Abelian group G admits compact topology if, and only if, it is of the form*:

$$G=D+\sum_{u}^{*} J_n\Big/\sum_{j}^{*} J_j,$$

where D is the maximal divisible subgroup of G and fulfils conditions of Theorem 1, *J_i and J_j are groups of p-adic integers and each projection $\pi_{i,j}$ of the group J_j into J_i is an isomorphism on $\pi_{ij}(J_j)=0$.*

By $\sum^{*} G_t$ we denote the complete direct sum of the groups G_t.

Proofs of the theorems (which will be published in Fundamenta Mathematicae) make use of the duality theory and related results as presented in Pontriagin [4], S. Hartman and Ryll-Nardzewski [1] and J. Łoś [3].

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MATHÉMATIQUE

Propriétés des intégrales de l'équation parabolique générale

par

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Présenté par K. KURATOWSKI le 28 Avril 1956

Considérons l'équation aux dérivées partielles du type parabolique

$$\hat{\Psi}(u)=\sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}(A,t)\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta}+\sum_{\alpha=1}^{n} b_\alpha(A,t)\frac{\partial u}{\partial x_\alpha}+c(A,t)\,u-\frac{\partial u}{\partial t}=0, \tag{1}$$

où les coefficients $a_{\alpha\beta}(A,t)$, $b_\alpha(A,t)$ et $c(A,t)$ sont des fonctions définies dans la région fermée

$$A(x_1,\ldots,x_n)\in \Omega+S, \quad (0\leqslant t\leqslant T), \tag{2}$$

où Ω est un domaine borné dans l'espace euclidien à $n\geqslant 2$ dimensions et limité par la surface fermée S.

Admettons quatre hypothèses suivantes:

1. La forme quadratique

$$\sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}(A,t)\,X_\alpha X_\beta \tag{3}$$

est *définie positive* dans la région (2).

2. Les coefficients $a_{\alpha\beta}(A,t)$ satisfont dans la région (2) à la condition d'Hölder de la forme

$$|a_{\alpha\beta}(A,t)-a_{\alpha\beta}(A_1,t_1)|<k[r^h_{AA_1}+|t-t_1|^{h'}], \tag{4}$$

k étant une constante positive, h et h' étant des constantes positives ne dépassant pas 1 et r_{AA_1} désignant la distance entre les points A et A_1.

3. Les coefficients $b_\alpha(A,t)$ et $c(A,t)$ sont des fonctions continues de l'ensemble de variables $(x_1,\ldots,x_n;\,t)$ dans la région (2) et satisfont à la condition d'Hölder par rapport aux variables spatiales de la forme

$$\begin{aligned} &|b_\alpha(A,t)-b_\alpha(A_1,t)|<k' r^h_{AA_1},\\ &|c(A,t)-c(A_1,t)|<k'' r^h_{AA_1}. \end{aligned} \tag{5}$$

4. La surface S satisfait aux conditions de Liapounov dont l'une, concernant l'angle $\Delta(P,Q)$ entre deux normales aux points P et Q, est de la forme

$$\Delta(P,Q) < \text{const.}\, r_{PQ}^{\varkappa} \quad (0<\varkappa\leqslant 1). \tag{6}$$

Nous avons démontré dans le travail [1] que sous les hypothèses 1 — 4 il existe une solution fondamentale de l'équation (1) et qu'elle est donnée par la formule

$$\Gamma(A,t;B,\tau)=w^{B,\tau}(A,t;B,\tau)+\int_{\tau}^{t}\iiint_{\Omega'(M)} w^{M,\xi}(A.t;M,\xi)\,\Phi(M,\xi;B,\tau)\,dv_M d\xi, \tag{7}$$

où

$$w^{M,\tau}(A,t;B,\tau)=(t-\tau)^{-n/2}\exp\left[-\frac{\vartheta^{M,\tau}(A,B)}{4(t-\tau)}\right],$$
$$\vartheta^{M,\tau}(A,B)=\sum_{\alpha,\beta=1}^{n} a^{\alpha\beta}(M,\tau)(x_\alpha-\xi_\alpha)(x_\beta-\xi_\beta); \tag{8}$$

ici $a^{\alpha\beta}(M,\tau)$ désignent les éléments de la matrice *inverse* à la matrice $\|a_{\alpha\beta}(M,\tau)\|$ et Ω' désigne un domaine arbitraire mesurable contenant à l'intérieur l'ensemble $\Omega+S$. Les coefficients $a_{\alpha\beta}$, b_α et c sont supposés prolongés au domaine Ω' d'une façon quelconque, pourvu que les hypothèses 1, 2, et 3 soient maintenues.

La fonction Φ qui figure dans la formule (7) est la solution d'une équation singulière de Volterra (voir [1], formules 13, 14, 16 et 17).

La fonction (7) est définie pour tout couple $A(x_1,\ldots,x_n)$, $B(\xi_1,\ldots,\xi_n)$ de points du domaine Ω' et pour $0\leqslant\tau<t\leqslant T$.

La communication présente contient les résultats de nos recherches concernant les propriétés des quelques intégrales de l'équation (1).

Définition 1. On appelle *potentiel de la couche simple relatif à l'équation* (1) l'intégrale de surface

$$U(A,t)=\int_{0}^{t}\iint_{S}\Gamma(A,t;Q,\tau)\varphi(Q,\tau)\,d\sigma_Q d\tau, \tag{9}$$

où la fonction $\varphi(Q,\tau)$ — dite la *densité* de la couche — est définie dans la région

$$Q\in S, \quad (0\leqslant\tau\leqslant T). \tag{10}$$

Définition 2. On appelle *potentiel de la charge spatiale relatif à l'équation* (1), l'intégrale de volume

$$V(A,t)=\int_{0}^{t}\iiint_{\Omega}\Gamma(A,t;B,\tau)\,\varrho(B,\tau)\,dv_B d\tau, \tag{11}$$

où la fonction $\varrho(B,\tau)$ — dite la *densité* de la charge — est définie dans la région

$$B \in \Omega, \qquad (0 \leqslant \tau \leqslant T). \tag{12}$$

THÉORÈME 1. *Si la densité $\varphi(Q,\tau)$ est une fonction continue dans la région* (10), *la dérivée transversale*

$$\frac{dU(A,t)}{dT_P} = \sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}(A,t)\cos(N_P,x_\beta)\,U_{x_\alpha}(A,t) \tag{13}$$

du potentiel de la couche simple (9) *au point intérieur A du domaine Ω tend uniformément à la limite suivante:*

$$\lim_{A\to P}\frac{dU(A,t)}{dT_P} = \tag{14}$$

$$= -\frac{(2\sqrt{\pi})^n}{2\sqrt{\det|a^{\alpha\beta}(P,t)|}}\,\varphi(P,t) + \int_0^t\int_S\int \frac{d\Gamma(P,t;Q,\tau)}{dT_P}\,\varphi(Q,\tau)\,d\sigma_Q\,d\tau,$$

lorsque le point A tend au point P de la surface S et que $0<t\leqslant T$.

La dérivée transversale de la solution fondamentale au même point P de la surface S vérifie l'inégalité aux singularités faibles de la forme

$$\left|\frac{d}{dT_P}[\Gamma(P,t;Q,\tau)]\right| < \frac{\text{const}}{(t-\tau)^{\mu^*}}\cdot\frac{1}{r_{PQ}^{n+1-2\mu^*-\varkappa_1}}, \tag{15}$$

où μ^* est un nombre intérieur de l'intervalle $\left(1-\frac{\varkappa_1}{2},1\right)$ et $\varkappa_1 = \inf(h,2h',\varkappa)$.

THÉORÈME 2. *Si la densité $\varphi(Q,\tau)$ est une fonction bornée et intégrable dans la région* (10), *le potentiel de la couche simple* (9) *satisfait dans la région fermée*

$$A \in \Omega + S \qquad (0 \leqslant t \leqslant T) \tag{16}$$

à la condition d'Hölder de la forme

$$|U(A,t)-U(A_1,t_1)| < \text{const.}\,\sup|\varphi|\,[r_{AA_1}^{\varrho} + |t-t_1|^{\theta'/2}], \tag{17}$$

où θ et θ' sont des nombres positifs arbitraires, inférieurs à 1.

THÉORÈME 3. *Si la densité $\varphi(Q,\tau)$ est une fonction bornée et intégrable dans la région* (10), *l'intégrale*

$$H(P,t) = \int_0^t\int_S\int \frac{d}{dT_p}[\Gamma(P,t;Q,\tau)]\varphi(Q,\tau)\,d\sigma_Q\,d\tau, \tag{18}$$

qui est une sommande du membre droit de l'égalité (14), *est une fonction définie dans la région $[P\in S;\ 0\leqslant t\leqslant T]$ et satisfait à la condition d'Hölder de la forme*

$$|H(P,t)-H(P_1,t_1)|\,\text{const.} < \sup|\varphi|\,[r_{PP_1}^{\theta\varkappa_1} + |t-t_1|^{\theta'\varkappa_1/3}], \tag{19}$$

où θ et θ' sont des nombres positifs arbitraires, inférieurs à l'unité.

THÉORÈME 4. *Si la densité* $\varrho(B,\tau)$ *est une fonction bornée et intégrable dans la région* (12), *le potentiel de la charge spatiale* (11) *satisfait par rapport à la variable* t *à la condition d'Hölder de la forme*

$$|V(A,t)-V(A,t_1)| < \text{const}\,|t-t_1|^{\theta} \tag{20}$$

et les premières dérivées spatiales de ce potentiel satisfont à la condition d'Hölder de la forme

$$|V_{x_\alpha}(A,t)-V_{x_\alpha}(A_1,t_1)| < \text{const}\,[r^{\theta}_{AA_1}+|t-t_1|^{\theta'/2}], \tag{21}$$

où $0<\theta<1$ *et* $0<\theta'<1$.

THÉORÈME 5. *Si la fonction* $\varrho(B,\tau)$ *est bornée et continue dans la région* (12), *l'intégrale de Poisson-Weierstrass généralisée*

$$J(A,t,\tau)=\int\int\int_{\Omega} \Gamma(A,t;B,\tau)\,\varrho(B,\tau)\,dv_B \qquad (0<\tau<t\leqslant T) \tag{22}$$

tend à la limite

$$\lim_{\tau\to t} J(A,t,\tau)=\frac{(2\sqrt{\pi})^n}{\sqrt{\det|a^{\alpha\beta}(A,t)}}\,\varrho(A,t) \tag{23}$$

uniformément dans tout domaine fermé Ω^* *situé à l'intérieur du domaine* Ω.

Les démonstrations des théorèmes 1–5 seront publiées dans un travail qui paraîtra sous le même titre dans Annales Polonici Mathematici.

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OUVRAGES CITÉS

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Cl. III — Vol. IV, No. 7, 1956

MATHEMATICS

An Application of Kulikov's Basic Subgroups in the Theory of Abelian Mixed Groups

by

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Presented by K. KURATOWSKI on April 7, 1956

In the theory of mixed Abelian groups, many unsolved problems deal with conditions under which the torsion subgroup T of an Abelian group G is a direct summand of G. Obviously the (torsion-free) factor group $G/T=H$ is of great importance in this research. It may, for instance, be asked under what conditions on T and H, T is a direct summand of G, (the (T,H)-problem), or under what conditions on H, T is a direct summand of G (the (H)-problem). In the (T,H)-problem, G may be an arbitrary extension of the torsion group T with the torsion-free group H, where T and H are fixed groups. In the (H)-problem, only group H is fixed: G may be an arbitrary extension of an arbitrary torsion group T with H.

The purpose of this paper is to present an application of Kulikov's basic subgroup to the reduction of both problems, (T,H) and (H). We shall demonstrate the following

THEOREM 1. *T is a direct summand of every group G with $G/T=H$ if, and only if, the basic subgroup B of T is a direct summand of every group G' with $G'/B=H$.*

THEOREM 2. *For every extension G of an arbitrary torsion group T with fixed (torsion-free) group H to be decomposable in a direct sum $G=T+H$, it is sufficient that every extension G of an arbitrary discrete sum C of finite cyclic groups with H, be decomposable in a direct sum $G=C+H$.*

By the basic subgroup B of a torsion group T we mean a subgroup of T with the following properties [3]:

1° B is a serving subgroup of T;

2° B is a direct sum of cyclic groups;

3° the factor group T/B is a divisible group.

We see that in reality, Theorem 1 is a reduction of the (T,H)-problem, and Theorem 2 a reduction of the (H)-problem. Moreover, we see

from property 2^0 of the basic subgroup that Theorem 2 results from Theorem 1. Therefore, we shall prove only Theorem 1.

Proof of necessity. Let H be represented as a factor group of some Abelian free group S; $H=S/R$. We prove that:

(1) if every homomorphism of R in the torsion group T is induced by a homomorphism of S in T, then every homomorphism of R in the basic subgroup B of T is also induced by a homomorphism of S in B.

Let h_0 be an arbitrary homomorphism of R in B, and ε an endomorphism of T on B (for the existence of such an endomorphism see [4]). We define the homomorphism h of R in T as follows:

For elements a_ν of basis of R we put

$$h(a_\nu)=t_\nu,$$

where t_ν is an arbitrary element of B with $\varepsilon(t_\nu)=h_0(a_\nu)$.

By assumption, the homomorphism h is induced by a homomorphism h^* of S in T. It is evident that the homomorphism h_0 is induced by the homomorphism $h_0^*=\varepsilon h^*$ and $h_0^*(S)\subset B$.

Our proposition follows from (1) and from the following theorem of S. Eilenberg and S. MacLane [2]. The group of Abelian extension of Abelian group A with Abelian group B, which is represented in the form S/R (S-Abelian free), is isomorphic to the factor group $\mathcal{H}/\mathcal{H}_0$, where $\mathcal{H}$ is the group of all homomorphisms of R in A, and $\mathcal{H}_0$ consists of those homomorphisms which are induced by homomorphisms of S in A.

Proof of sufficiency. Let G be an arbitrary Abelian extension of torsion group T with torsion-free factor group H and B a basic subgroup of T. By 3^0 the torsion subgroup $\bar{T}$ of the factor group $\bar{G}=G/B$ is divisible. Hence, $\bar{G}$ can be represented as a direct sum of subgroups $\bar{T}$ and $\bar{H}$ *), which is isomorphic to H.

$$\bar{G}=\bar{T}+\bar{H} \quad \text{and} \quad \bar{H}\cong H. \tag{2}$$

Let G' be the subgroup of G containing B and such that $G'/B=\bar{H}$. Group G' is an extension of B with H. From the assumption that the group B is a direct summand of every G' with $G'/B\cong H$, it follows that

$$G'=B+H'. \tag{3}$$

We shall see that $G=T+H'$, which is sufficient for the proof. In fact, since H' is torsion-free we have $T\cap H'=\{0\}$. It remains to prove that every element g of G belongs to the group $T+H'$. Let $g\in G$; by (2) the coset $\bar{g}$ modulo B can be represented in the form $\bar{g}=\bar{t}+\bar{h}$, where $t\in T$ and $h\in G'$. Then $g=t+h+b$ ($b\in B$). But $h\in G'$ and by (3) we have $h=b'+h'$ and $g=(t+b+b')+h'\in T+H'$.

*) A divisible group is a direct summand of every group which it contains [1].

Remark. It is easy to see that if G is an arbitrary extension of a discrete sum C of finite cyclic group with torsion free group H, then there exists a decomposition

$$G=G'+C'',$$

where $C''\subset C$ and the power of the torsion subgroup C' of G' satisfies the condition $\overline{\overline{C}}'\leqslant\overline{\overline{H}}$.

From this we obtain the following generalisation of Theorem 2:

THEOREM 3. *For every extension G of an arbitrary torsion group T with fixed (torsion-free) group H to be decomposable in a direct sum $G=T+H$, it is sufficient that every extension G of an arbitrary discrete sum C of finite cyclic groups with H, where $\overline{\overline{C}}\leqslant\overline{\overline{H}}$, be decomposable into a direct sum $G=C+H$.*

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MATHEMATICS

On Free Subgroups of Topological Groups

by

S. BALCERZYK and JAN MYCIELSKI

Presented by W. SIERPIŃSKI on April 21, 1956

The rank of a free group F (or of an Abelian free group F) is the cardinality of a set of free generators of F.

It was proved by W. Sierpiński [3] (cf. also J. de Groot and T. Dekker [1]) that the group of rotations of the sphere $(x^2+y^2+z^2=1)$ contains a free subgroup of the rank $2^{\aleph_0}$. According to M. Kuranishi [2] every connected semi-simple Lie group contains a free subgroup of the rank 2.

In the present note we are announcing some generalisations of these statements. The proofs will be published in Fundamenta Mathematicae.

THEOREM 1. *Every compact (= bicompact), connected, non-Abelian group contains a free subgroup of the rank* $2^{\aleph_0}$.

THEOREM 2. *Every locally compact, connected, non-solvable group contains a free subgroup of the rank* $2^{\aleph_0}$.

Note, that a solvable group cannot contain any free group of a rank >1 since such groups are not solvable.

THEOREM 3. *Every locally compact, non-0-dimensional, Abelian group contains a free Abelian subgroup of the rank* $2^{\aleph_0}$.

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MATHEMATICS

On the Decompositions of Euclidean Spaces

by

JAN MYCIELSKI

Presented by W. SIERPIŃSKI on April 5, 1956

It is the purpose of this paper to state certain theorems, the proofs of which will be published in Fundamenta Mathematicae. Theorems 1 and 2 are solutions of problems suggested to the author by T. Dekker.

THEOREM 1. *There exists a free group G of the rank $2^{\aleph_0}$ of sense-preserving isometries of the Euclidean space E^3 such that no isometry in G, different from the identity, has a fixed point (i. e. for any $\varphi \in G-(e)$ and $p \in E^3$ there is $\varphi(p) \neq p$).*

Let M and N be abstract sets of potency $\leqslant 2^{\aleph_0}$. Theorem 1 implies

THEOREM 2. *For any system of congruences of the form*

$$(*) \qquad \sum_{\nu \in P_\mu} A_\nu \simeq \sum_{\nu \in Q_\mu} A_\nu \qquad \mu \in M,$$

*$\{P_\mu\}_{\mu \in M}$ and $\{Q_\mu\}_{\mu \in M}$ are families of subsets of N, different from 0 and N *) there exists a decomposition of the space E^3 into $\overline{\overline{N}}$ disjoint sets $\{A_\nu\}_{\nu \in N}$ which satisfy system (*) and such that all these congruences are realisable by a sense-preserving isometry.*

In general there exists a decomposition of a metric space E into disjoint pieces $\{A_\nu\}_{\nu \in N}$ satisfying the system (*) if there exists a free group of isometries of E, of rank $\overline{\overline{M}}$, without fixed points. Some less restrictive conditions for a space E which imply the existence of such decompositions were found by T. Dekker (paper not yet published).

The existence of such decompositions of the sphere $(x^2+y^2+z^2=1)$ is known [1], [2], [3], but here sense-reversing isometries are necessary in general (the condition for sense-preserving isometries is also well examined in this case, [2] Theorem (R_2)). This result on S^2 obviously implies the same result for the space E^3 without one point. The de-

*) Here and subsequently we do not suppose that $\mu_1 \neq \mu_2$ implies $P_{\mu_1} \neq P_{\mu_2}$ or $Q_{\mu_1} \neq Q_{\mu_2}$,

composition statement for E^n with $n \geqslant 3$ obviously follows from Theorem 2. For the line E^1 the decompositions in general do not exist ([5], p. 56), neither do they exist for the plane E^2 as was recently proved by T. Dekker. However the following theorem holds

THEOREM 3. *For any system of relations of the form*

$$\sum_{\nu \in P_\mu} A_\nu \underset{2}{=} \sum_{\nu \in Q_\mu} A_\nu \text{*)} \quad \mu \in M$$

($\{P_\mu\}_{\mu \in M}$ *and* $\{Q_\mu\}_{\mu \in M}$ *are families of non-empty subsets of* N), *there exists on the plane a family of* $\overline{\overline{N}}$ *disjoint non-empty sets* $\{A_\nu\}_{\nu \in N}$ *satisfying this system.*

For results related to this theorem see Sierpiński [4].

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[5] — *On the congruence of sets and their equivalence by finite decomposition*, Lucknow, 1954.

*) $\underset{2}{=}$ denotes congruence of sets by decomposition into two parts. For an exact explanation see, for instance, [5] p. 24.

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MATHEMATICS

On the Congruence of Sets

by

JAN MYCIELSKI

Presented by W. SIERPIŃSKI on April 5, 1956

1. Let M and M' be analytic manifolds *). A mapping f of M into M' is called *analytic* if the local co-ordinates of a point $f(p) \in M'$ are analytic functions of the local co-ordinates of the point $p \in M$.

THEOREM 1. *Let M be an analytic manifold and G a free group of rank 2 of analytic homeomorphisms of M onto itself. Then there exists a countable set $E \subset M$, such that for each finite set $F \subset E$ there exists a homeomorphism $h_F \in G$, such that*

$$h_F(E) = E - F.$$

From this theorem we can deduce for several geometric spaces the existence of a countable set E in this space, such that

$$E - F \simeq E \quad \text{for every finite set} \quad F \subset E.$$ **)

For example, such is the case for the spaces:

(1) the sphere $(x^2+y^2+z^2=1)$ ***),

(2) the elliptic plane,

(3) the hyperbolic plane ****),

because for these spaces there exist free groups of rank 2 of distance-preserving analytic homeomorphisms:

The existence of such a free group of rotations of the sphere is well known. The existence of such groups of isometrics of the elliptic and hyperbolic plane (with some additional properties) was recently proved by T. Dekker (as yet unpublished).

---

*) The manifolds considered are real and connected.

**) $\simeq$ denotes congruence of sets.

***) This was first stated in [4] and proved in [3].

****) This was proved independently by Professor T. Viola (to be published in the Atti della Accademia Nazionale dei Lincei).

As a general fact, the application of Theorem 1 requires the existence of certain free groups. Several general conditions for a topological group implying the existence of a free subgroup are known; in particular every compact, connected, non Abelian group contains a free subgroup of rank $2^{\aleph_0}$ [1] (and then of rank 2 also). This applies for the groups of isometries of the spaces (1) and (2).

2. We introduce the following condition on a family of sets $\boldsymbol{F}$ and a cardinal number $\mathfrak{m}$:

(A) $\mathfrak{m}$ is infinite and $\boldsymbol{F}$ can be well ordered in a sequence $\{F_\xi\}$, such that for every F_{ξ_0} there is $\sum_{\xi \leqslant \xi_0} \overline{\overline{F_\xi}} < \mathfrak{m}$.

THEOREM 2. *Let G be a group and $[M]$ a free subgroup of G with a set M of free generators. Let $\boldsymbol{F}$ be a family of subsets of G. We suppose that $\boldsymbol{F}$ and $\overline{\overline{M}}$ satisfy (A).*

Then there exists a set $E \subset G$ such that for any set $F \in \boldsymbol{F}$ there exists a $\varphi \in M$, such that

$$\varphi E = (E - F) + (F - E) \text{ *).}$$

As an application of this theorem we can prove the following

THEOREM 3. *A free group G of the rank $2^{\aleph_0}$ [of any rank $\mathfrak{m} \geqslant 2$] contains a set E, such that for any sets A, $B \subset C$ which are at most denumerable [which are finite] there exists a $\varphi \in G$, such that*

$$\varphi E = (E - A) + B.$$

The algebraic lemma given in my papers [3], [4] is an immediate consequence of Theorem 2.

3. We give here a geometrical theorem analogous to the algebraical theorems of the preceding section and Theorem 1.

THEOREM 4. *There exists on the sphere $S(x^2 + y^2 + z^2 = 1)$ [in the Euclidean space E^3] a set E such that*

$$E \simeq (E - A) + B \text{ **)}$$

for any sets $A, B \subset S$ [$A, B \subset E^3$] which are at most denumerable.

This theorem generalises Theorem 2 of my papers [3] and [4]. Some generalisations of Theorem 4 for analytic manifolds (in the direction of Theorem 1) can be made, but only with complicated hypotheses concerning the manifold.

4. The proofs of these theorems will be given in the paper *On sets with strange isometrical properties (II)*, which is being prepared for Fundamenta Mathematicae.

*) φE denotes the set of all elements of G which are of the form $\varphi\eta$ where $\eta \in E$. $+$ and $-$ have the usual set theoretical meaning.

**) $\simeq$ denotes congruence of sets realisable by a sense-preserving isometry.

In the proofs of Theorems 1 and 4, the following lemma of S. Balcerzyk and the author [2] plays an essential role:

LEMMA. *Let M and M' be analytic manifolds, and $\{f_t\}_{t\in T}$ and $\{g_t\}_{t\in T}$ be families of analytic mappings of M into M'.*

If for any $t \in T$ there exists a $p_t \in M$, such that $f_t(p_t) \neq g_t(p_t)$, and if $\overline{\overline{T}} < 2^{\aleph_0}$, then the set

$$\sum_{t\in T} \underset{p\in M}{E} [f_t(p) = g_t(p)]$$

is a border set in M.

Other applications of this lemma given in [5] are also used.

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MATHÉMATIQUE

Sur la croissance de la fonction opérationnelle $\exp(-s^a\lambda)$

par

J. MIKUSIŃSKI

Présenté par K. KURATOWSKI le 3 Mai 1956

1. La fonction opérationnelle $\exp(-s^a\lambda)$, où s est l'opérateur différentiel, joue un rôle fondamental dans la méthode opérationnelle de la résolution des équations aux dérivées partielles [1]. Dans le cas $0<a<1$, cette fonction peut être interprétée comme fonction ordinaire de deux variables, à savoir

$$F_a(\lambda,t)=\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\exp(zt-z^a\lambda)\,dz \qquad (\lambda>0,\ t\geqslant 0), \tag{1}$$

où z^a est la branche qui admet des valeurs positives sur la partie positive de l'axe réel des z. Pour $t>0$, cette fonction est positive. Il est bien connu que, pour $a=\frac{1}{2}$, l'intégrale (1) se laisse évaluer effectivement, ce qui conduit à l'expression

$$F_{1/2}(\lambda,t)=\frac{\lambda}{2\sqrt{\pi t^3}}\exp\left(-\frac{\lambda^2}{4t}\right), \tag{2}$$

importante dans la théorie de l'équation de la chaleur.

Lorsque $a\neq\frac{1}{2}$, l'intégrale (1) ne se laisse pas évaluer explicitement, on peut cependant estimer directement son ordre de croissance. Dans cette note nous démontrerons que

$$F_a(\lambda,t)<\left(\frac{a\lambda}{t}\right)^{\frac{1}{1-a}}\exp\left[-(1-a)\,a^{\frac{a}{1-a}}\left(\frac{\lambda}{t^a}\right)^{\frac{1}{1-a}}\right] \quad \text{pour} \quad \frac{\lambda}{t^a}>\frac{2}{a^5}. \tag{3}$$

Dans le cas où $a=\frac{1}{2}$, la dernière inégalité devient

$$F_{1/2}(\lambda,t)<\left(\frac{\lambda}{2t}\right)^2\exp\left(-\frac{\lambda^2}{4t}\right) \quad \text{pour} \quad \frac{\lambda}{\sqrt{t}}>64;$$

on voit que l'estimation de l'ordre de croissance est exacte, abstraction faite du coefficient $\left(\frac{\lambda}{2t}\right)^2$.

2. Le chemin d'intégration dans (1) est l'axe imaginaire. Ce chemin peut être remplacé par le contour représenté sur la figure ci-jointe, car l'intégrale le long du demi-cercle se laisse écrire

$$\frac{k}{2\pi}\int_{-\pi/2}^{\pi/2}\exp\left[k e^{i\varphi}t - k^{\alpha}e^{i\alpha\varphi}\lambda\right]e^{i\varphi}\,d\varphi \tag{4}$$

et l'on voit qu'elle disparaît pour $k\to 0$.

En posant $k=\left(\frac{\lambda}{\omega t}\right)^{\frac{1}{1-\alpha}}$, le module de (4) est inférieur à

$$\frac{k}{2\pi}\int_{-\pi/2}^{\pi/2}\exp\left[\left(\frac{\lambda}{\omega t^{\alpha}}\right)^{\frac{1}{1-\alpha}}(\cos\varphi-\omega\cos\alpha\varphi)\right]d\varphi.$$

La fonction $-\sin\varphi+\omega\alpha\sin\alpha\varphi$ est négative dans l'intervalle $0<\varphi<\pi/2$ et positive dans l'intervalle $-\pi/2<\varphi<0$, lorsque

$$\omega\leqslant\frac{1}{\alpha\sin\frac{\alpha\pi}{2}}. \tag{5}$$

Il s'ensuit que, dans le cas (5), la fonction $\cos\varphi-\omega\cos\alpha\varphi$ atteint son maximum pour $\varphi=0$; la valeur de ce maximum est évidemment $1-\omega$. L'intégrale (4) est donc inférieure à

$$\frac{k}{2}\exp\left[\omega^{-\frac{1}{1-\alpha}}(1-\omega)\left(\frac{\lambda}{t^{\alpha}}\right)^{\frac{1}{1-\alpha}}\right]. \tag{6}$$

Le produit $\omega^{-\frac{1}{1-\alpha}}(1-\omega)$ atteint son minimum pour $\omega=1/\alpha$. Donc, en posant cette valeur dans (6), nous obtiendrons la meilleure approximation de (4):

$$\frac{1}{2}\alpha^{\frac{1}{1-\alpha}}\lambda^{\frac{1}{1/\alpha}}\left(\frac{\lambda}{t^{\alpha}}\right)^{\frac{1}{\alpha(1-\alpha)}}\exp\left[-(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\left(\frac{\lambda}{t^{\alpha}}\right)^{\frac{1}{1-\alpha}}\right]. \tag{7}$$

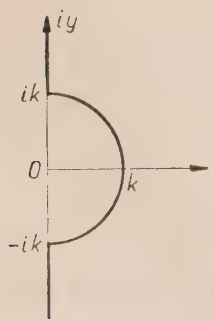

Fig. 1

Considérons maintenant l'intégrale (1) le long du rayon au-dessus du demi-cercle. Son module est inférieur à

$$\frac{1}{2\pi}\int_{k}^{\infty}\exp\left(-\lambda\varrho\cos\frac{\pi\alpha}{2}\right)d\varrho.$$

En substituant $\sigma=\lambda\varrho^{\alpha}$, la dernière intégrale devient

$$\frac{1}{2\pi\alpha\lambda^{1/\alpha}}\int_{\lambda k^{\alpha}}^{\infty}\sigma^{\frac{1}{\alpha}-1}\exp\left(-\sigma\cos\frac{\pi\alpha}{2}\right)d\sigma.$$

Si $\sigma \geqslant 2/\alpha^4$, la fonction sous le signe de l'intégrale est inférieure à $\exp[-\sigma(1-\alpha)]$. Si

$$\frac{\lambda}{t^\alpha} \geqslant \frac{2}{\alpha^5}, \tag{8}$$

on a $\lambda k^\alpha \geqslant 2/\alpha^4$, et l'intégrale est inférieure à

$$\frac{1}{2\pi\alpha(1-\alpha)\lambda^{1/\alpha}} \exp\left[-(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\left(\frac{\lambda}{t^\alpha}\right)^{\frac{1}{1-\alpha}}\right]. \tag{9}$$

La même estimation (9) a lieu pour l'intégrale le long du rayon au-dessous du demi-cercle. L'intégrale (1) est donc inférieure à la double valeur (9) augmentée de (7). En vertu des inégalités

$$\frac{1}{\pi^\alpha(1-\alpha)}\left(\frac{\alpha^3}{2}\right)^{\frac{1}{\alpha(1-\alpha)}} < \frac{1}{2}\alpha^{\frac{1}{1-\alpha}} \quad \text{et} \quad \left(\frac{2}{\alpha^3}\right)^{\frac{1}{\alpha(1-\alpha)}} < \left(\frac{\lambda}{t^\alpha}\right)^{\frac{1}{\alpha(1-\alpha)}},$$

cette intégrale est inférieure à la double valeur (7). C'est ce qu'exprime l'inégalité (3).

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ASTRONOMIE

Der Einfluss der Bewegung der Dunkelmaterie auf den Lichtwechsel der Sterne

von

K. RUDNICKI

Vorgelegt von W. DZIEWULSKI am 10. April 1956

Da die allgemeine Lösung des Problems über den Einfluss der Bewegung der Dunkelmaterie auf den Lichtwechsel der Sterne zu kompliziert wäre, wurde ein vereinfachertes Modell betrachtet. Es wurde angenommen, dass die Dunkelmaterie ausschliesslich in der Form von diskreten kugelsymmetrischen Nebeln mit zufälliger radialer Verteilung der Dichte vorkommt.

Infolge der relativen Bewegung eines Nebels schneidet die Gerade Beobachter-Stern den Nebel in einer Fläche eines kleinen Kreises. Auf Grund elementarer Geometrieverfahren kann man die Mittel der Zeitableitung der Sterngrösse m abschätzen. Und zwar

$$\overline{\left|\frac{dm}{dt}\right|} = 2V_{\perp}\, a_0,$$

wo $V_{\perp}$ — die Geschwindigkeitskomponente des Nebels senkrecht zur Geraden Beobachter-Stern, und a_0 — den Mittebwert der Absorption pro Längeneinheit innerhalb des Nebels unter Beachtung aller möglichen Schnittweisen bezeichnen. a_0 ist auf Grund der Beobachtungen von vielen Nebeln als ein Mittel $a_0 = \overline{\left(\frac{A}{l}\right)}$ zu bestimmen, wo A — die totale Absorption innerhalb eines Nebels und l — die Nebeldicke sind. $\overline{\left|\frac{dm}{dt}\right|}$ ist von der Grösse des Nebels, sowie von der totalen Absorption innerhalb des Nebels unabhängig.

Wenn sich längs der Geraden Beobachter-Stern mehrere Nebel befinden, erhält man eine allgemeine Formel nach Ausdrücken von $V_{\perp}$, durch die Grössen, die unmittelbar von Beobachtungen oder Abschätzungen erhalten werden können:

$$\overline{\frac{dm}{dt}} = \pm 2a_0 \sqrt{\sum_i \left(V_t^2 \frac{R_i^2}{R_0^2} + V_\odot^2 \cos^2 \lambda + 2V_t V_\odot \frac{R_i}{R_0} \cos \lambda \cos \vartheta + \sigma_t \right)}$$

wo:

V_t — die Tangentialgeschwindigkeit des Sterns,
R_i — die Entfernung eines einzelnen Nebels,
R_0 — die Entfernung des Sterns,
$V_\odot$ — die Apeksgeschwindigkeit der Sonne,
λ — die Winkelentfernung vom Apeks,
ϑ — der Winkel zwischen der Pekuliar- und der Paralaktischen Komponente der Eigenbewegung des Sterns,
σ_t — die Dispersion der Tangentialgeschwindigkeit des Nebels

bedeuten.

Angenommen, dass ϑ willkrüliche Werte annimmt und die Verteilung der Nebel längs der Gerade Beobachter-Stern gleichmässig ist, gelingt man zu einem vereinfacherten Ausdruck, und zwar:

$$\overline{\frac{dm}{dt}} = \pm 2a_0 \sqrt{n} \cdot \sqrt{\frac{1}{3} V_t^2 + V_\odot^2 \cos^2 \lambda + \sigma_t^2},$$

wo n die Zahl der Nebel ist, die durch den Lichtstrahl des Sterns zerschneidet werden. Für die Zonen der Milchstrasse, die etwa 90^0 vom Apeks entfernt sind (Perseus und Triangulum Austrinum — Norma), beim Einsetzen von $V_t = V_\odot = 20$ km./sek., $\sigma_t = 5$ km./sek., $n = 9$ (einer Entfernung von etwa 3/4 kps entsprechend) und $a_0 = 0^m,12$/ps, erhält man

$$\overline{\frac{dm}{dt}} = \pm 7 \cdot 10^{-14} \text{ magnitudo/sek.}$$

Bei Schnelläufern sind grössere Veränderungen des Lichtes zu erwarten, welche eine mittlere Grösse von etwa $0^m,002$ pro 10 Jahre erreichen können. Es ist möglich solch ein Effekt mit Hilfe relativer elektrophotometrischen Messungen, die alle 10-20 Jahre zu wiederholen sind, zu bestätigen.

UNIVERSITÄTSSTERNWARTE, WARSZAWA

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ASTRONOMY

On the Motion of Red Dwarfs with Emission Spectra

by

J. SMAK

Presented by W. DZIEWULSKI on May 11, 1956

These objects have recently attracted considerable attention of numerous investigators, since it is suspected that red dwarfs with emission spectra are somewhat young.

This opinion, based chiefly on studies of the space distribution of such stars, and their close connection with interstellar matter, has been expressed recently by, among others, V. A. Amborzoumian [1] and G. Haro [2].

J. Delhaye [3], in 1953, determined the dispersion of z components of velocities of these stars, and found that its value for ordinary red dwarfs is considerably higher than that of red dwarfs with emission spectra; this supports the opinion referred to above. In the same year, J. E. Ejnasto [4] determined the velocity-ellipsoid for both groups of stars, using the proper motion data contained in A. Vyssotsky's catalogue [5]. Yet the selection factor in the material concerning the red dwarfs with emission spectra might influence Ejnasto's results to such a degree as to throw doubt on their reliability.

The aim of the present paper is, on the ground of material free from the selection factor, to re-determine the velocity-ellipsoid of red dwarfs with emission spectra, in order to establish the kinematic differences between this group of stars and red dwarfs with absorption spectra.

Such material is contained in E. R. Dyer's [6] paper concerning radial velocities of red dwarfs; this is free from selection, as may be seen from Fig. 1, where histograms of radial velocities for stars with different luminosities are given. Thus the writer had to his disposal, after rejecting certain items which appeared to be components of binary systems with unknown elements, 67 objects with emission spectra and 91 ordinary red dwarfs.

Histograms of radial velocities in Fig. 2. show distinctly that the dispersion of observed radial velocities of red dwarfs with emission spectra is considerably smaller than that of ordinary red dwarfs.

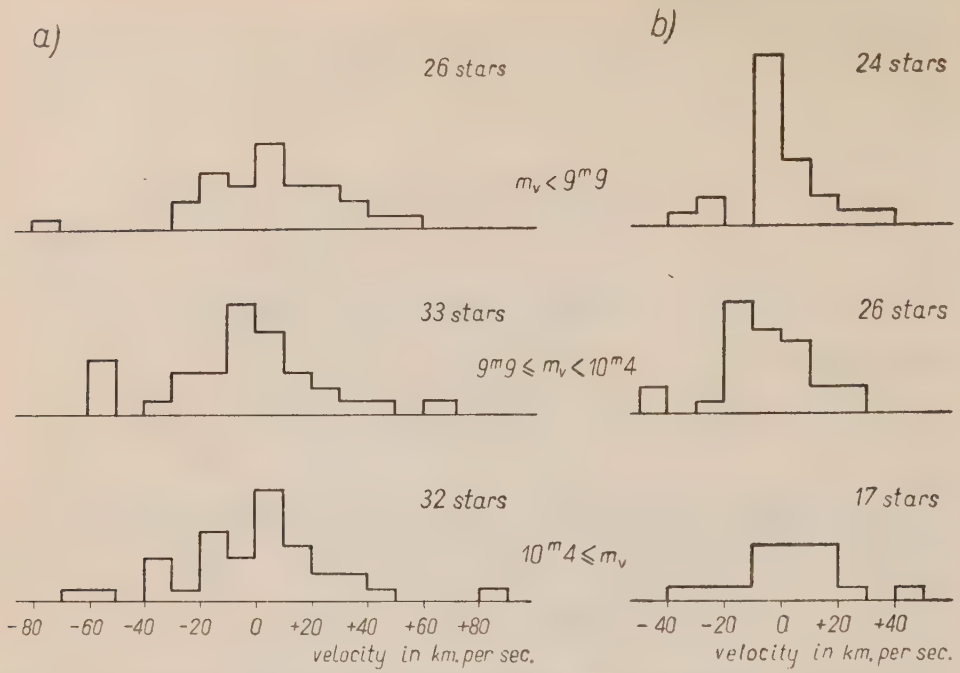

Fig. 1. Histograms of radial velocities for stars with different luminosities (a – for red dwarfs with absorption spectra, b – for red dwarfs with emission spectra).

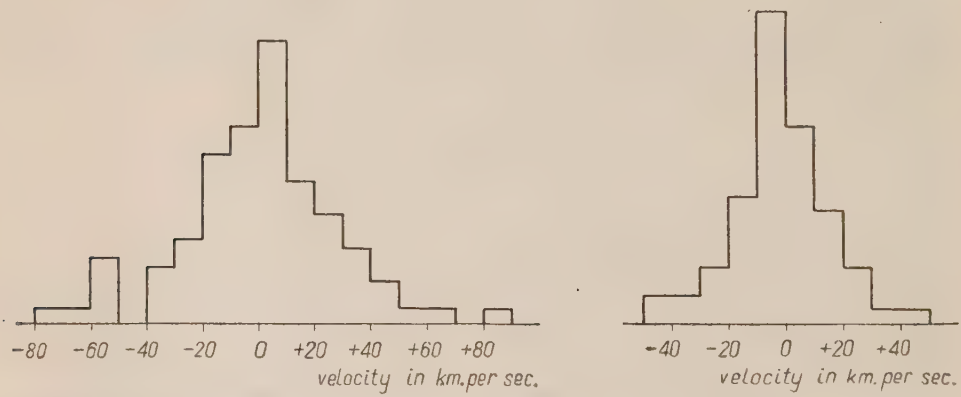

Fig. 2. Histograms of radial velocities for all ordinary red dwarfs (left) and for all red dwarfs with emission spectra (right).

First, the writer determined the motion of the Sun in relation to the centroids of both groups of stars. Results are shown in Table I. The relative velocity of the two centroids amounts to

$$21.1 \pm 13.0 \text{ km./sec.}$$

Thus may be established the real difference in the velocities of the two centroids together with the fact that the velocity of centroids of

TABLE I

	$V_\odot$ km./sec.	Galactic co-ordinates of apex		K km./sec.
		l_0	b_0	
Red dwarfs with emission spectra	2.7±6.3	347°±80°	11°±65°	−2.6±8.6
Ordinary red dwarfs	22.7±9.1	36°±20°	17°±17°	+2.0±7.4

dwarfs with emission spectra is practically the same as the peculiar motion of the Sun.

Table II gives the velocity-ellipsoids of the two groups of stars determined by the method of Charlier [7].

TABLE II

		Semi-axes of ellipsoid		
		I	II	III
Emission spectra red dwarfs	σ (in km./sec.)	27.5±1.5	16.1±2.3	12.9±4.2
	σ/σ_I	1.00	0.58±0.07	0.47±0.09
	l	346°	284°5	70°5
	b	−2°	+21°5	+58°5
Ordinary red dwarfs	σ (in km./sec.)	46.4±1.0	23.3±1.3	16.1±3.9
	σ/σ_I	1.00	0.51±0.08	0.35±0.03
	l	21°5	103°	244°
	b	+4°5	+46°	+41°5

The dispersion of the velocities of emission spectra stars is smaller than that of ordinary dwarfs; this is in agreement with the results obtained by Delhaye and Ejnasto. It is worth while observing that the present results resemble that obtained by Ejnasto [4] for two groups of dM stars (under the assumption that the two ellipsoids are rotational ones). Table III presents the results obtained by Ejnasto. Thus it may be concluded that the first group of dM stars in the Ejnasto investigation is almost identical with red dwarfs with emission spectra.

TABLE III

Group	Great semi-axes km./sec.	Small semi-axes km./sec.
I	24.0± 2.3	14.4±1.4
II	49.1±11.8	29.5±7.1

The small dispersion of velocities applying to these stars, together with some other physical characteristics, suggests that red dwarfs with emission spectra belong to the group of stars which are somewhat young. These objects are probably responsible for certain kinematic inhomogenities in the stars forming the lower part of the main sequence in the H-R diagram noticed by Ejnasto [4].

In conclusion, I should like to express my gratitude to Professor W. Zonn for his continuous interest and guidance, to A. Kruszewski for having suggested this problem, and for many interesting discussions, and to K. Serkowski for valuable advice.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 7, 1956

THEORETICAL PHYSICS

New Method of Solving the Diffraction Problem for a Dipole Field

by

R. TEISSEYRE

Presented by L. INFELD on March 31, 1956

1. In a recent paper [8] a method was given of solving the diffraction problem of a perfectly conducting wedge. The method consists in a generalisation of the Senior method [6] for the diffraction by a conducting half-plane. In another paper [9], the general solutions for the wedge problem were presented.

These solutions are obtained by means of the method given by the author in a previous paper [8]. We now present a new method for treating this problem. The method is, in general, analogous to that of Senior, but is much more direct and simple. The mathematical difficulties which undeniably existed when applying methods proposed in [6] and [8] have now been overcome.

2. From the identity for the boundary conditions for E_3, Π_3^e and H_3, Π_3^m (the x_3 axis is coincident with the edge, Π_3^e — Hertz vector of an electric dipole, Π_3^m — Hertz vector of a magnetic dipole) it follows that for Π_3^e, Π_3^m we may take the Sommerfeld solutions for the spherical waves of an electric and magnetic type [4], [6].

From the physical meaning follows the possibility of building the electromagnetic field of an arbitrary dipole from the fields of the Π_3^e and Π_3^m dipoles [6].

We can find the proper superposition by considering the vacuum field.

3. We take the co-ordinate system [8]:

$$(r, \theta, z) = (x_k) \text{ — co-ordinate of field,}$$

$$(r_0, \theta_0, z_0) = (x_k^0) \text{ — co-ordinate of dipole.}$$

The Hertz vector of a dipole in vacuum can be represented in the following manner:

$$(1)\qquad \frac{e^{-ikR}}{kR} = \boldsymbol{S} e^{ik[r\cos(\theta_1-\alpha)\cos\beta - r_0\cos(\theta_0-\alpha)\cos\beta-(z-z_0)\sin\beta]} = \boldsymbol{S} e^{ik\varphi_l(x_l-x_l^0)},$$

where operator $\boldsymbol{S}$ equals

$$\frac{1}{2\pi i}\int_{-i\infty}^{\pi/2}\cos\beta\, d\beta \int_{-\pi}^{\pi} d\alpha,$$

$$\varphi_1 = \cos\alpha\cos\beta, \qquad \varphi_2 = \sin\alpha\cos\beta, \qquad \varphi_3 = -\sin\beta.$$

Hence, the electromagnetic field of the Π_3^e dipole with an amplitude A_3^e can be expressed as follows:

$$(2)\qquad E_k^e = A_3^e \boldsymbol{S}\mathcal{E}_k^e(\alpha,\beta), \qquad H_k^e = A_3^e \boldsymbol{S}\mathcal{H}_k^e(\alpha,\beta),$$

where

$$(2a)\qquad \begin{aligned} \mathcal{E}_k^e &= (-k^2\varphi_k\varphi_3 + k^2\delta_{k3})\, e^{ik\varphi_l(x_l-x_l^0)},\\ \mathcal{H}_k^e &= -k^2\varepsilon_{kr3}\varphi_r e^{ik\varphi_l(x_l-x_l^0)}. \end{aligned}$$

Analogously, for Π_3^m

$$(3)\qquad E_k^m = A_3^m \boldsymbol{S}\mathcal{E}_k^m(\alpha,\beta), \qquad H_k^m = A_3^m \boldsymbol{S}\mathcal{H}_k^m(\alpha,\beta),$$

where

$$(3a)\qquad \begin{aligned} \mathcal{E}_k^m &= -k^2\varepsilon_{kr3}\varphi_r e^{ik\varphi_l(x_l-x_l^0)},\\ \mathcal{H}_k^m &= (k^2\varphi_k\varphi_3 - k^2\delta_{k3})\, e^{ik\varphi_l(x_l-x_l^0)}. \end{aligned}$$

For the field of an arbitrary dipole we build the superposition with the coefficients $\lambda(\alpha,\beta)$, $\mu(\alpha,\beta)$:

$$(4)\qquad E_k^e = A_s^e \boldsymbol{S}(\lambda_s^e\mathcal{E}_k^e + \mu_s^e\mathcal{E}_k^m), \qquad H_k^e = A_s^e \boldsymbol{S}(\lambda_s^e\mathcal{H}_s^e + \mu_s^e\mathcal{H}_k^m).$$

and

$$(5)\qquad E_k^m = A_s^m \boldsymbol{S}(\lambda_s^m\mathcal{E}_k^e + \mu_s^m\mathcal{E}_k^m), \qquad H_k^m = A_s^m \boldsymbol{S}(\lambda_s^m\mathcal{H}_k^e + \mu_s^m\mathcal{H}_k^m).$$

By comparison of (4) and (5) with the well known expressions for a vacuum field we obtain the coefficients

$$(4a)\qquad \lambda_s^e = \frac{1}{\cos^2\beta}(\varphi_s\varphi_3 - \delta_{s3}), \qquad \mu_s^e = \frac{1}{\cos^2\beta}\varepsilon_{sr3}\varphi_r,$$

$$(5a)\qquad \lambda_s^m = \frac{-1}{\cos^2\beta}\varepsilon_{sr3}\varphi_r, \qquad \mu_s^m = \frac{1}{\cos^2\beta}(\varphi_s\varphi_3 - \delta_{s3}).$$

4. We now pass to the wedge problem. For Π_3^e and Π_3^m we take the Sommerfeld solutions, as mentioned above.

$$(6)\qquad \Pi_3^e = \frac{i}{2\varkappa}\int_A d\gamma \frac{e^{-ik\sqrt{r^2+r_0^2-2rr_0\cos(\theta-\gamma)+(z-z_0)^2}}}{\sqrt{r^2+r_0^2-2rr_0\cos(\theta-\gamma)+(z-z_0)^2}} \frac{\sin\frac{\pi}{\varkappa}\theta_0}{\cos\frac{\pi}{\varkappa}\gamma-\cos\frac{\pi}{\varkappa}\theta_0},$$

$$(7)\qquad \Pi_3^m = \frac{i}{2\varkappa}\int_A d\gamma \frac{e^{-ik\sqrt{r^2+r_0^2-2rr_0\cos(\theta-\gamma)+(z-z_0)^2}}}{\sqrt{r^2+r_0^2-2rr_0\cos(\theta-\gamma)+(z-z_0)^2}}\left(\frac{\sin\frac{\pi}{\varkappa}\gamma}{\cos\frac{\pi}{\varkappa}\gamma-\cos\frac{\pi}{\varkappa}\theta_0}-i\right),$$

where the A-path is given in Fig. 1.

The electromagnetic field of these dipoles can be represented in a form analogous to (2) and (3). The general solutions can be constructed by (4) and (5).

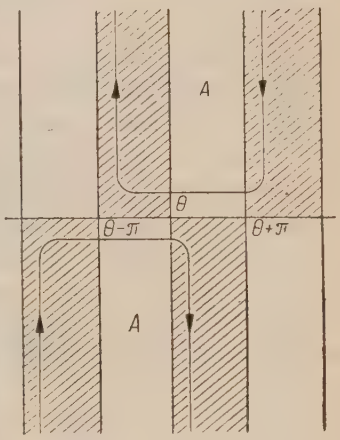

Fig. 1

5. It is convenient for further discussion to transform the original Sommerfeld solutions (6), (7) so that the integration path is deformed into the steepest descent path.

In the first step we can get the integrals along the contour C [8] and then along the steepest descent path $P(0)$ through point $\gamma=0$ [6]. The poles swept over by the deforming the path, represent the so-called geometrical optic field.

In this way one gets:

$$(8)\qquad \Pi_3^e = \delta(\theta<\pi+\theta_0)\frac{e^{-ikR}}{kR}-\delta(\theta<\pi-\theta_0)\frac{e^{-ikS}}{kS}-\delta(2\varkappa<\pi+\theta+\theta_0)\frac{e^{-ikT}}{kT}+$$
$$+\frac{\sin\frac{\pi^2}{\varkappa}}{4\varkappa i}\int_{P(0)} d\gamma \sum_{l=1,2}\left[(-)^l\frac{e^{-ikR_p}}{kR_\gamma}\frac{1}{\sin\frac{\pi}{2\varkappa}(\gamma+\varphi_l+\pi)\sin\frac{\pi}{2\varkappa}(\gamma+\varphi_l-\pi)}\right],$$

$$(9)\qquad \Pi_3^m = \delta(\theta<\pi+\theta_0)\frac{e^{-ikR}}{kR}+\delta(\theta<\pi-\theta_0)\frac{e^{-ikS}}{kS}+\delta(2\varkappa<\pi+\theta+\theta_0)\frac{e^{-ikT}}{kT}+$$
$$+\frac{\sin\frac{\pi^2}{\varkappa}}{4\varkappa i}\int_{P(0)} d\gamma \sum_{l=1,2}\left[\frac{e^{-ikR_p}}{kR_\gamma}\frac{1}{\sin\frac{\pi}{2\varkappa}(\gamma+\varphi_l+\pi)\sin\frac{\pi}{2\varkappa}(\gamma+\varphi_l-\pi)}\right],$$

where $\delta(a<b)=1$ for $a<b$, $\delta(a<b)=0$ for $a>b$; R, S, T are the distances from the dipole and the dipole's images in the upper and lower surface of the wedge;

$$R_\gamma=\sqrt{r^2+r+2rr_0\cos\gamma+(z-z_0)^2},\qquad \varphi_1=\theta_0+\theta,\qquad \varphi_2=\theta_0-\theta.$$

All the above terms involve the function e^{-iku}/ku, hence representation (2), (3) can be obtained.

The general solutions [9] obtained by (4) and (5), have the diffraction field built from the integrals of the form:

$$(10)\qquad A_s \frac{\sin\frac{\pi^2}{\varkappa}}{4\varkappa i}\int\limits_{P(0)} d\gamma\,\Omega_{ks}(\gamma)\frac{e^{-ikR_p}}{kR_\gamma}\frac{1}{\sin\frac{\pi}{2\varkappa}(\gamma+\varphi_l+\pi)\sin\frac{\pi}{2\varkappa}(\gamma+\varphi_l-\pi)},$$

where $\Omega_{ks}(\gamma)$ is a certain differential expression depending continuously on γ [9].

6. For the short waves we can give the approximation of formula (10).

Since kR is large, only the small finite tangential element of the path in the point $\gamma=0$ is of importance ($\gamma=\delta e^{i\pi/4}$, $-\varepsilon<\delta<\varepsilon$).

Now we must take into account the shadow boundary defined by the discontinuities of the geometrical optic field, or by the zero points of the denominator in (10).

Far from the shadow boundary all functions are continuous and we can put $\gamma=0$ everywhere, except in the exponential factor which changes rapidly. Assuming $\varepsilon\sqrt{\frac{krr_0}{2}}$ to be large, we obtain

$$(11)\qquad A_s\Omega_{ks}\frac{\sin\frac{\pi^2}{\varkappa}}{4\varkappa i}\sqrt{\frac{2\pi}{krr_0}}\frac{e^{-ikR_1}}{kR_1}\frac{1}{\sin\frac{\pi}{2\varkappa}(\varphi_l+\pi)\sin\frac{\pi}{2\varkappa}(\varphi_l-\pi)};$$

where $R_1=\sqrt{(r+r_0)^2+(z-z_0)^2}$.

Now we shall discuss the field near the shadow boundary ($\theta_0+\theta=\pi$, $\theta_0+\theta=2\varkappa-\pi$, $\theta_0-\theta=-\pi$). By analogy to the former case we can take the integral on a small tangential element

$$(12)\qquad A_s\Omega_{ks}\frac{\sin\frac{\pi^2}{\varkappa}}{4\varkappa i}\frac{e^{i\pi/4}}{\sin\frac{\pi}{2\varkappa}(\varphi_l-\pi)}\int\limits_{-\varepsilon}^{\varepsilon} d\delta\,\frac{e^{-ikR_p}}{kR_\gamma}\frac{1}{\sin\frac{\pi}{2\varkappa}(\gamma+\varphi_l+\pi)}.$$

We assume that the sine-function under the integral is nearly zero, that is, its argument is near the shadow boundary. For δ small we have [6]

$$\frac{e^{-ikR_p}}{kR_\gamma}=\frac{1}{\sqrt{2\pi}}\int\limits_0^{a-i\infty}\frac{ds}{s^{3/2}}e^{s/2-\frac{k^2R_1^2}{2s}+\frac{ik^2rr_0\delta^2}{2s}}.$$

After performing some simple manipulations, (12) takes on the form

$$(13)\qquad A_s\Omega_{ks}\frac{\sin\frac{\pi^2}{\varkappa}}{\sin\frac{\pi}{2\varkappa}(\varphi_l-\pi)}\frac{e^{i\pi/4}}{\pi\sqrt{2\pi}}\eta\int_0^{a-i\infty}\frac{ds}{s^{3/2}}e^{s/2-\frac{k^2R_1^2}{2s}}\int_0^{\varepsilon k\sqrt{rr_0}}d\tau\frac{e^{i\frac{\tau^2}{2s}}}{\tau^2+i\eta^2},$$

where $\eta=\frac{2\varkappa}{\pi}k\sqrt{rr_0}\sin\frac{\pi}{2\varkappa}(\varphi_l+\pi)$.

For $\varepsilon k\sqrt{rr_0}$ large, we have

$$(14)\qquad \eta\int_0^{\infty}d\tau\frac{e^{i\frac{\tau^2}{2s}}}{\tau^2+i\eta^2}=\mp ie^{i\pi/4}\sqrt{\frac{\pi}{2s}}e^{\eta^2/2s}\int_{\pm\eta}^{\infty}d\lambda e^{-\frac{\lambda^2}{2s}},$$

where the upper signs refer to $\eta>0$, the lower to $\eta<0$. Introducing (14) into (13) and taking into account [6]

$$\int_0^{a-i\infty}\frac{ds}{s^2}e^{s/2-\frac{k^2R_1^2-\eta^2+\lambda^2}{2s}}=\frac{\pi}{i}\frac{H_1^{(2)}\left(\sqrt{k^2R_1^2-\eta^2+\lambda^2}\right)}{\sqrt{k^2R_1^2-\eta^2+\lambda^2}},$$

we obtain

$$(15)\qquad A_s\Omega_{ks}\frac{\sin\frac{\pi^2}{\varkappa}}{\sin\frac{\pi}{2\varkappa}(\varphi_l-\pi)}\frac{1}{2i}\int_{\pm\eta}^{\infty}d\lambda\frac{H_1^{(2)}\left(\sqrt{k^2R_1^2-\eta^2+\lambda^2}\right)}{\sqrt{k^2R_1^2-\eta^2+\lambda^2}}.$$

For kR_1 large we can take the asymptotic expansion for the $H_1^{(2)}$ function. We finally obtain

$$(16)\qquad \pm A_s\Omega_{ks}\frac{\sin\frac{\pi^2}{\varkappa}}{\sin\frac{\pi}{2\varkappa}(\varphi_l-\pi)}\frac{e^{i\pi/4}}{\sqrt{\pi}}\frac{e^{-ikR_1+iu^2}}{kR_1}\int_{\pm u}^{\infty}d\mu\, e^{-i\mu^2},$$

where $u=\frac{\varkappa}{\pi}\sqrt{\frac{2krr_0}{R_1}}\sin\frac{\pi}{2\varkappa}(\varphi_l+\pi)$; the upper signs refer to $u>0$, the lower to $u<0$.

We remember that on the shadow boundaries $u=0$ and $R=S=T=R_1$. We see, therefore, that passing through the shadow boundary, integral (10) has the discontinuity $\pm\frac{1}{2}A_s\Omega_{ks}\frac{e^{-ikR_1}}{kR_1}$, which cancels the discontinuities of the geometrical optic field.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 7, 1956

On a Generalisation of the Einstein-Infeld Approximation Method

by

A. TRAUTMAN

Presented by L. INFELD on April 9, 1956

1. The new approximation method formulated first by Einstein, Infeld and Hoffman [1], and developed in further works of Einstein, Infeld [2]-[5], Robertson [6], Papapetrou [7] and others (for references see [8]) is successfully used to derive the equations of motion in the general relativity theory. The method is based on expanding all functions (the metric tensor $g_{\alpha\beta}$, the energy-momentum tensor $T_{\alpha\beta}$) into Taylor series in the parameter $\lambda=1/c$ ($c=$ velocity of light) and on introducing an auxiliary time $\tau=\lambda x^0$. In the papers quoted above, it was assumed that these series contain terms corresponding either to odd or to even powers of λ, e. g., $g_{00}=\sum_{l=0}^{\infty}\lambda^{2l}\,\underset{2l}{g}_{00}$ (as stated by Infeld [5]). A further assumption was that $\underset{0}{g}_{\alpha\beta}$ has the Galilean values

$$\underset{0}{g}_{00}=1, \qquad \underset{0}{g}_{0k}=0, \qquad \underset{0}{g}_{mn}=-\delta_{mn} \tag{1}$$

(Greek indices run from 0 to 3, Latin – from 1 to 3). In this paper it is shown that the first of these assumptions can be removed and the second easily generalised. An application of the modified theory will be given in a subsequent paper.

2. We use the conventional notation of the theory of relativity (e. g., summation convention). The derivatives with respect to x^r we denote by a stroke $|$, that with respect to τ or x^k by a comma, viz.:

$$\frac{\partial\varphi}{\partial x_a}=\varphi_{|a}, \qquad \frac{\partial\varphi}{\partial x^0}=\varphi_{|0}=\lambda\varphi_{,0}=\lambda\frac{\partial\varphi}{\partial\tau}, \qquad \frac{\partial\varphi}{\partial x^k}=\varphi_{|k}=\varphi_{,k}.$$

(a) when written under a letter denotes $\delta_{0\alpha}$.

The gravitational field equations we shall write in the form

$$R_{\alpha\beta}=\varkappa(T_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}T), \tag{2}$$

where

$$R_{\alpha\beta}=\tfrac{1}{2}g^{\mu\nu}(g_{\alpha\mu,\beta\nu}+g_{\beta\mu,\alpha\nu}-g_{\alpha\beta,\mu\nu}-g_{\mu\nu,\alpha\beta})+g^{\mu\nu}g^{\sigma\varrho}(\Gamma_{\nu\alpha\sigma}\Gamma_{\mu\varrho\beta}-\Gamma_{\nu\alpha\beta}\Gamma_{\mu\varrho\sigma}), \tag{3}$$

$T_{\alpha\beta}$ denotes the energy-momentum tensor, $T=T^{\alpha}_{\alpha}$, and

$$\Gamma_{\alpha\beta\gamma}=\tfrac{1}{2}(g_{\alpha\beta,\gamma}+g_{\gamma\alpha,\beta}-g_{\beta\gamma,\alpha}).$$

We assume that in the region of the x^{ν}-space, where we look for the field, we can develop $g_{\alpha\beta}$, $g^{\alpha\beta}$ and $T_{\alpha\beta}$ into power series:

$$g_{\alpha\beta}=\sum_{l=0}^{\infty}\lambda^{l}\underset{l}{g_{\alpha\beta}},\qquad g^{\alpha\beta}=\sum_{l=0}^{\infty}\lambda^{l}\underset{l}{g^{\alpha\beta}},\qquad T_{\alpha\beta}=\sum_{l=0}^{\infty}\lambda^{l}\underset{l}{T_{\alpha\beta}}. \tag{4}$$

From $g_{\alpha\nu}g^{\nu\beta}=\delta^{\beta}_{\alpha}$ we obtain

$$\underset{0}{g_{\alpha\nu}}\,\underset{0}{g^{\nu\beta}}=\delta^{\beta}_{\alpha}, \tag{5}$$

$$\sum_{k=0}^{n}\underset{k}{g_{\alpha\nu}}\,\underset{n-k}{g^{\nu\beta}}=0,\qquad n=1,2,\ldots \tag{6}$$

From (5) and (6) we can evaluate $\underset{l}{g^{\alpha\beta}}$ as a function of the $\underset{0}{g_{\alpha\beta}}, \underset{1}{g_{\alpha\beta}}, \ldots, \underset{l}{g_{\alpha\beta}}$ and $\underset{0}{g^{\alpha\beta}}$ (or vice-versa). The result is

$$\underset{l}{g^{\alpha\beta}}=\sum(-1)^{p}\,\underset{0}{g^{\alpha\nu_1}}\,\underset{k_1}{g_{\nu_1\nu_2}}\,\underset{0}{g^{\nu_2\nu_3}}\,\underset{k_2}{g_{\nu_3\nu_4}}\cdots\underset{k_p}{g_{\nu_{2p-1}\nu_{2p}}}\,\underset{0}{g^{\nu_{2p}\beta}}. \tag{7}$$

The sum in (7) is to be taken over all (ordered) sets of positive integers $k_1,k_2,\ldots,k_p$ such that $k_1+k_2+\ldots+k_p=l$. Assuming for $R_{\alpha\beta}$ an expansion similar to that of $T_{\alpha\beta}$, we have

$$\begin{aligned}\underset{l}{R_{\alpha\beta}}=\tfrac{1}{2}\sum_{k=0}^{l}\underset{l-k}{g^{\mu\nu}}\,(&\underset{k-(\beta)-(\nu)}{g_{\alpha\mu,\beta\nu}}+\underset{k-(\alpha)-(\nu)}{g_{\beta\mu,\alpha\nu}}-\underset{k-(\mu)-(\nu)}{g_{\alpha\beta,\mu\nu}}-\underset{k-(\alpha)-(\beta)}{g_{\mu\nu,\alpha\beta}})+\\+\tfrac{1}{4}\sum_{k=0}^{l}\sum_{j=0}^{k}\sum_{i=0}^{j}&\underset{l-k}{g^{\mu\nu}}\,\underset{k-j}{g^{\sigma\varrho}}\,[(\underset{j-i-(\sigma)}{g_{\nu\alpha,\sigma}}+\underset{j-i-(\alpha)}{g_{\sigma\nu,\alpha}}-\underset{j-i-(\nu)}{g_{\alpha\sigma,\nu}})\,(\underset{i-(\varrho)}{g_{\mu\beta,\varrho}}+\underset{i-(\beta)}{g_{\varrho\mu,\beta}}-\underset{i-(\mu)}{g_{\varrho\beta,\mu}})+\\-&(\underset{j-i-(\beta)}{g_{\nu\alpha,\beta}}+\underset{j-i-(\alpha)}{g_{\nu\beta,\alpha}}-\underset{j-i-(\nu)}{g_{\alpha\beta,\nu}})\,(\underset{i-(\varrho)}{g_{\mu\varrho,\sigma}}+\underset{i-(\varrho)}{g_{\mu\sigma,\varrho}}-\underset{i-(\mu)}{g_{\varrho\sigma,\mu}})].\end{aligned} \tag{8}$$

Equation (8) gives, together with (7), an explicit expression of $\underset{l}{R_{\alpha\beta}}$ as a function of the $\underset{k}{g^{\alpha\beta}}$'s and their derivatives. The Einstein equations (2) are equivalent to a set of equations:

$$\underset{l}{R_{\alpha\beta}}=\varkappa\Big(\underset{l}{T_{\alpha\beta}}-\tfrac{1}{2}\sum_{k=0}^{l}\sum_{j=0}^{k}\underset{l-k}{g_{\alpha\beta}}\,\underset{k-j}{g^{\mu\nu}}\,\underset{j}{T_{\mu\nu}}\Big),\qquad l=0,1,2,\ldots \tag{9.$l.\alpha\beta$}$$

3. Let us assume that in our co-ordinate system the tensor $\underset{0}{g_{\alpha\beta}}$ has the form:

$$\underset{0}{g_{\alpha\beta}}=\left(\begin{array}{c|c}1 & 0\\ \hline 0 & \underset{0}{g_{ik}}\end{array}\right), \tag{10}$$

where $\underset{0}{g_{ik}} = \underset{0}{g_{ik}}(x^\nu)$, and that the x^k-space with the metric $\underset{0}{g_{ik}}$ is (for each $x^0 = \text{const.}$) Euclidean. This means that we assume a flat space-time in the zero-order approximation, and use inertial but in general non-cartesian co-ordinate systems.

From these assumptions we have $\underset{0}{R_{\alpha\beta}} = 0$ ($\underset{0}{R_{ik}} = 0$ follows from the fact that $\underset{0}{g_{ik}}$ is Euclidean), which implies, by virtue of (9.0. $\alpha\beta$), that $\underset{0}{T_{\alpha\beta}} = 0$. The set of functions $\underset{l}{g_{mn}}$ (or $\underset{l}{R_{mn}}$) $(m, n = 1, 2, 3)$ forms a covariant tensor with respect to transformations of the 3-dimensional space: $x^k \to x^{k'}$ (not involving λ). Likewise $\underset{l}{g_{0n}}$ (or $\underset{l}{R_{0n}}$) is a 3-vector and $\underset{l}{g_{00}}$ (or $\underset{l}{R_{00}}$) is a scalar. We shall denote by a semicolon the covariant differentiation in the x^k-space with metric $\underset{0}{g_{ik}}$. From (8) we get

$$\underset{l}{R_{00}} = -\tfrac{1}{2}\underset{0}{g^{mn}}\left(\underset{l}{g_{00,mn}} - \underset{0}{g^{rs}}\underset{0}{g_{mr,s}}\underset{l}{g_{00,n}} + \tfrac{1}{2}\underset{0}{g^{rs}}\underset{0}{g_{rs,m}}\underset{l}{g_{00,n}}\right) + \underset{l}{R'_{00}}$$

$$= -\tfrac{1}{2}\underset{0}{g^{mn}}\underset{l}{g_{00;mn}} + \underset{l}{R'_{00}} = \tfrac{1}{2}\underset{l}{\Delta g_{00}} + \underset{l}{R'_{00}}, \tag{11}$$

$$\underset{l}{R_{0k}} = \tfrac{1}{2}\underset{0}{g^{mn}}\left(\underset{l}{g_{0m;kn}} - \underset{l}{g_{0k;mn}}\right) + \underset{l}{R'_{0k}}, \tag{12}$$

$$\underset{l}{R_{ik}} = \tfrac{1}{2}\underset{0}{g^{mn}}\left(\underset{l}{g_{im;kn}} - \underset{l}{g_{ik;mn}} + \underset{l}{g_{nk;mi}} - \underset{l}{g_{mn;ki}}\right) - \tfrac{1}{2}\underset{l}{g_{00;ik}} + \underset{l}{R'_{ik}}. \tag{13}$$

The functions $\underset{l}{R'_{\alpha\beta}}$ depend only on $\underset{0}{g_{\mu\nu}}, \underset{1}{g_{\mu\nu}}, \ldots, \underset{l-1}{g_{\mu\nu}}$ and their derivatives, and can be explicitily evaluated from (8). The right-hand side of (9.l. $\alpha\beta$) does not involve $\underset{l}{g_{\alpha\beta}}$, because $\underset{0}{T_{\alpha\beta}} = 0$. Thus we see that $\underset{l}{g_{\alpha\beta}}$ enters in (9. l. $\alpha\beta$) linearly and only through the terms written explicitly in (11)-(13). An essential fact is that the equations (9. l. $\alpha\beta$) do not involve the time derivatives of $\underset{l}{g_{\alpha\beta}}$.

Equation (9. l. 00) has a unique solution providing the x^ν-space is pseudo-Euclidean at infinity. This is not, however, the case with equations (9. l. ak). If $\underset{l}{g_{\alpha\beta}}$ is a solution of (9. l. $\alpha\beta$), then $\underset{l}{g^*_{\alpha\beta}}$ defined by the relations

$$\begin{aligned} \underset{l}{g^*_{00}} &= \underset{l}{g_{00}}, \\ \underset{l}{g^*_{0k}} &= \underset{l}{g_{0k}} + \underset{l}{\varphi_{0;k}} \\ \underset{l}{g^*_{ik}} &= \underset{l}{g_{ik}} + \underset{l}{\varphi_{i;k}} + \underset{l}{\varphi_{k;i}}, \end{aligned} \tag{14}$$

($\underset{l}{\varphi_a}$ — arbitrary functions) is also a solution of these equations. This can be easily seen from (11)-(13). The existence of several solutions of the field equations is in conformity with the general principle of relativity;

the transformation $\underset{l}{g_{\alpha\beta}} \to \underset{l}{g^*_{\alpha\beta}}$ can be regarded as resulting from a co-ordinate transformation [9].

Assuming certain conditions for the co-ordinate system, e. g.,

$$\underset{0}{g^{pq}}\underset{l}{g_{0p;q}}=0, \qquad \underset{0}{g^{pq}}(\underset{l}{g_{kp;q}}-\tfrac{1}{2}\underset{l}{g_{pq;k}})=0 \qquad (k=1,2,3), \tag{15}$$

we can simplify the expression for $R_{i\beta}$, which becomes, in the case of (15):

$$\begin{aligned} \underset{l}{R_{\alpha 0}} &= -\tfrac{1}{2}\,\underset{0}{g^{pq}}\underset{l}{g_{\alpha 0;pq}}+\underset{l}{R'_{\alpha 0}}, \\ \underset{l}{R_{ik}} &= -\tfrac{1}{2}\,\underset{0}{g^{pq}}\underset{l}{g_{ik;pq}}-\tfrac{1}{2}\,\underset{0}{g^{q}}\underset{l}{g_{00;ik}}+\underset{l}{R'_{ik}}. \end{aligned} \tag{16}$$

We conclude with a simple remark which can be useful in applying the method.

Let us take a flat x^ν-space (hence $T_{\alpha\beta}=0$); this means that there exists a Galilean co-ordinate system in which $g_{\alpha\beta}$ has the form (1). Let us now introduce a non-inertial co-ordinate system $x^{\nu'}$, defined by $\tau=\tau'$ (i. e. $x^0=x^{0'}$), $x^k=x^k(\tau', x^{k'})$. We have for $g_{\alpha'\beta'}$:

$$\begin{aligned} g_{0'0'} &= g_{\alpha\beta}\,x^\alpha_{|0'}\,x^\beta_{|0'}=g_{00}+\lambda^2\,g_{ik}\,x^i_{,0'}\,x^k_{,0'}=1-\lambda^2\delta_{ik}\,x^i_{,0'}\,x^k_{,0'}, \\ g_{0'k'} &= g_{\alpha\beta}\,x^\alpha_{|0'}\,x^\beta_{|k'}=-\lambda\delta_{ik}\,x^i_{,0'}\,x^k_{,k'}, \\ g_{i'k'} &= g_{\alpha\beta}\,x^\alpha_{|i}\,x^\beta_{|k'}=-\delta_{ik}\,x^i_{,i'}\,x^k_{,k'}. \end{aligned}$$

Thus, in a general non-inertial co-ordinate system, the metric tensor $g_{\alpha\beta}$ can be written $g_{\alpha\beta}=\underset{0}{g_{\alpha\beta}}+\lambda\underset{1}{g_{\alpha\beta}}+\lambda^2\underset{2}{g_{\alpha\beta}}$, where $\underset{0}{g_{\alpha\beta}}$ has the form (10),

$$\underset{1}{g_{\alpha\beta}}=\left(\begin{array}{c|c} 0 & \underset{1}{g_{0k}} \\ \hline \underset{1}{g_{i0}} & 0 \end{array}\right), \qquad \text{and} \qquad \underset{2}{g_{\alpha\beta}}=\left(\begin{array}{c|c} \underset{2}{g_{00}} & 0 \\ \hline 0 & 0 \end{array}\right).$$

I wish to express my thanks to Professor J. Plebański for suggesting the problem, and for many valuable discussions. I am also indebted to Professor L. Infeld for his kind interest in this work.

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BULLETIN DE L'ACADEMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV. No. 7. 1956

Solution of One-Body Problem by the Einstein-Infeld Approximation Method

by

A. TRAUTMAN

Presented by L. INFELD on April 9, 1956

In the previous paper [1] we presented a generalisation of the "new approximation method". In this paper we shall use that method to evaluate the gravitational field of a point mass, using the notation of [1]. We define $T_{\alpha\beta}$ so that $\varkappa = 8\pi k$, where $k = 6.67 \cdot 10^{-8}\text{cm.}^3\ \text{g}^{-1}\ \text{sec.}^{-2}$ (it is perhaps more usual to put $\varkappa = 8\pi\ kc^{-4}$). The energy-momentum tensor for a point mass will be represented by an expression involving the three-dimensional Dirac δ-function. This method of representing singularities was introduced by Infeld [2].

We assume

$$(1) \qquad \underset{0}{g}_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{pmatrix}$$

and denote

$$x^1 = r, \qquad x^2 = \vartheta, \qquad x^3 = \varphi.$$

Further,

$$(2) \qquad \underset{0}{T}{}^0_0 = \frac{1}{c^4} mc^2 \delta(\vec{r}) = m\lambda^2 \delta(\vec{r}) = \lambda^2 \underset{2}{T}{}^0_0,$$

$$\underset{0}{T}{}^\alpha_\beta = 0 \quad \text{if} \quad \alpha + \beta \neq 0.$$

We also assume that:

(3) *the metric $g_{\alpha\beta}$ is pseudo-Euclidean at infinity*;

(4) $\underset{l}{g}_{22} = \underset{l}{g}_{33} = \underset{l}{g}_{12} = \underset{l}{g}_{13} = \underset{l}{g}_{23} = 0$ and $\underset{l}{g}_{11,2} = \underset{l}{g}_{11,3} = 0$ ("spherical symmetry").

From the field equations (I. 9. 1. $\alpha\beta$) (we shall use the sign "I" when referring to the formulae of [1]), we obtain:

$$\Delta_1 g_{00} = 0, \tag{5}$$

$$g^{mn}_0 (g_{0m;kn} - g_{0k;mn})_1 = 0, \tag{6}$$

$$g^{mn}_0 (g_{im;kn} - g_{ik;mn} + g_{nk;mi} - g_{mn;ki})_1 - g_{00;ik} = 0. \tag{7}$$

Equation (5), together with (3), yields $g_{00} = 0$. Equations (6) and (7) have more than one solution (see (I. 14)), of which we choose the most simple: $g_{ak} = 0$ (this is equivalent to imposing certain new conditions on the co-ordinate system). A similar procedure will be applied to all homogeneous equations occurring in our further works. From (I. 9. 2. 00) we obtain the equation for g_{00}:

$$\tfrac{1}{2} \Delta_2 g_{00} = 8\pi k (T_{00} - \tfrac{1}{2} g_{00} T)_2 = 4\pi k g_{00} T^0_0 = 4\pi k m \delta(\vec{r}). \tag{8}$$

Hence,

$$g_{00} = -\frac{2km}{r} \overset{\text{def}}{=} -\psi. \tag{9}$$

Equations (I. 9. 2. $0k$) are in our case of the form (6), then $g_{0k} = 0$. Equation (I. 9. 2. 11) becomes

$$R_{11} = \tfrac{1}{2} g^{pq}_0 (2g_{1p;1q} - g_{11;pq} - g_{pq;11})_2 - \tfrac{1}{2} g_{00;11} = -4\pi k g_{11} T_2,$$

or

$$-\frac{1}{r} g_{11,1} = -\frac{2km}{r^3} + 4\pi k m \delta(\vec{r}). \tag{10}$$

Taking into account $r\delta(\vec{r}) = 0$, and (3), we get from (10):

$$g_{11} = -\frac{2km}{r}. \tag{11}$$

The equations for $g_{a\beta}$ are identical in form with those for $g_{a\beta}$; hence $g_{a\beta} = 0$. In the fourth-order approximation we have:

$$\begin{aligned} &R_{00} = \tfrac{1}{2} \Delta_4 g_{00} - \tfrac{1}{2} g^{mn}_2 g_{00,mn} - (g^{mn}_0 g^{rs}_2 + g^{mn}_2 g^{rs}_0) \Gamma_{2\,n00} \Gamma_{0\,mrs} + \\ &+ g^{\mu\nu}_0 g^{\sigma\varrho}_0 (\Gamma_{2\,\nu 0\sigma} \Gamma_{2\,\mu\varrho 0} - \Gamma_{2\,\nu 00} \Gamma_{2\,\mu\varrho\sigma}) = 4\pi k g_{00} T^0_0 = \tfrac{1}{2} g_{00} \Delta_2 g_{00}, \end{aligned} \tag{12}$$

from which it follows that

$$\tfrac{1}{2} \Delta_4 g_{00} + \tfrac{1}{2} g_{11} \Delta_2 g_{00} = \tfrac{1}{2} g_{00} \Delta_2 g_{00}, \tag{13}$$

and, in virtue of $\underset{2}{g}_{00}=\underset{2}{g}_{11}$, we obtain

$$\underset{4}{g}_{00}=0. \tag{14}$$

It should be noted that the expressions which cancel out from (13) are of the divergent type $\delta(\vec{r})r^{-1}$.

From (I. 9. 4. 11) we obtain an equation for $\underset{4}{g}_{11}$:

$$\underset{4}{g}_{11,1}=\frac{8k^2m^2}{r^3}-8\pi k^2m^2\delta(\vec{r}). \tag{15}$$

By means of the symbolic formulae

$$\delta(\vec{r})=\frac{\delta(r)}{2\pi r^2}, \qquad \delta(r)=\frac{1}{2}\frac{d}{dr}\frac{r}{|r|}, \tag{16}$$

we can solve (15):

$$\underset{4}{g}_{11,1}=-\frac{d}{dr}\frac{2k^2m^2}{r^2}-\frac{d}{dr}\left(\frac{2k^2m^2}{r^2}\frac{r}{|r|}\right).$$

Hence,

$$\underset{4}{g}_{11}=-\frac{2k^2m^2}{r^2}-\frac{2k^2m^2}{r^2}\frac{r}{|r|}=-\psi^2. \tag{17}$$

We shall now prove that for $l=2,3,\ldots$ we have

$$\underset{2l}{g}_{11}=-\psi^l, \qquad \underset{2l}{g}_{\alpha\beta}=0 \quad \text{if} \quad \alpha\beta\neq 1, \qquad \underset{2l-1}{g}_{\alpha\beta}=0. \tag{18}$$

This is true for $l=2$ from the previous results. Let us assume that (18) holds for $l<s$; then we get from (I. 7):

$$\underset{2}{g}^{11}=\psi, \qquad \underset{2l}{g}^{11}=0 \quad l=2,3,\ldots,s-1. \qquad \underset{2l}{g}^{00}=\psi^l, \quad l=1,2,\ldots,s-1. \tag{19}$$

From (I. 9. 2s. 00) we obtain the equation for $\underset{2s}{g}_{00}$:

$$\underset{2s}{R}_{00}=\tfrac{1}{2}\underset{2s}{\Delta g}_{00}+\underset{2s}{R'}_{00}=4\pi k\,\underset{2s-2}{g}_{00}\underset{2}{T}{}^0_0=0. \tag{20}$$

Evaluating $\underset{2s}{R'}_{00}$ and using (18) for $l<s$, we obtain $\underset{2s}{R'}_{00}=0$; thus $\underset{2s}{g}_{00}=0$. A somewhat troublesome calculus leads to the following equation for $\underset{2s}{g}_{11}$:

$$-\frac{1}{r}\underset{2s}{g}_{11,1}+\frac{s\psi^s}{r^2}=4\pi km\psi^{s-1}\delta(\vec{r}). \tag{21}$$

Solving (21) in a way similar to that of (15), we obtain

$$\underset{2s}{g}_{11}=-\frac{1}{2}\psi^s-\frac{1}{2}\psi^s\frac{|r|}{r}=-\psi^s. \tag{22}$$

The obvious equation $\underset{2s-1}{R'}_{\alpha\beta}=0$ implies $\underset{2s-1}{g}_{\alpha\beta}=0$; thus (18) holds for every $l>1$.

Now we are able to evaluate the metric tensor $g_{\alpha\beta}$:

$$g_{00} = \underset{0}{g_{00}} + \frac{1}{c^2}\underset{2}{g_{00}} = 1 - \frac{2km}{c^2 r},$$

$$g_{11} = \underset{0}{g_{11}} + \frac{1}{c^2}\underset{2}{g_{11}} + \frac{1}{c^4}\underset{4}{g_{11}} + \ldots = -\sum_{l=0}^{\infty}\frac{\psi^l}{c^{2l}} = -\frac{1}{1-\frac{2km}{c^2 r}}$$

$$\left(\text{convergent for } \frac{2km}{c^2} < r\right),$$

$$g_{0k} = 0, \qquad g_{12} = g_{23} = g_{13} = 0, \qquad g_{22} = \underset{0}{g_{22}}, \qquad g_{33} = \underset{0}{g_{33}}.$$

In this way is obtained the well-known Schwarzschild metric:

$$ds^2 = \left(1 - \frac{2km}{c^2 r}\right)dx_0^2 - r^2(\sin^2\vartheta\, d\varphi^2 + d\vartheta^2) - \frac{dr^2}{1-\frac{2km}{c^2 r}}.$$

The same result can be obtained by expanding $g_{\alpha\beta}$ into a series in $c = \lambda^{-1}$. Let us define $k' = kc^{-4}$ and $T'_{\alpha\beta} = c^4 T_{\alpha\beta}$. Thus, the right-hand side of the Einstein equations can be written:

$$8\pi k'(T'_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} T').$$

Expanding $T'_{\alpha\beta}$ into powers of c we get

$$T'^0_0 = c^4 T^0_0 = c^2 \underset{2}{T^0_0} = c^2 \underset{2}{T'^0_0}.$$

Hence, $\underset{2}{T'^0_0} = \underset{2}{T^0_0}$. We see that the solution obtained by an expansion in c follows from that obtained in this paper after applying a transformation: $k \to k'$, $c \to c^{-1}$. But the Schwarzschild metric involves k and c only, through a factor $k/c^2 \to k'c^2 = k/c^2$. The static nature of the field played an essential role in this argument.

I should like to express my thanks to Professors L. Infeld and J. Plebański for their kind interest in this work.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 7, 1956

THEORETICAL PHYSICS

On the Theory of the Electromagnetic Field in Moving Dielectrica

by

J. I. HORVÁTH

Presented by W. RUBINOWICZ on April 24, 1956

1. The relativistic theory of the electromagnetic field in its Hamiltonian formalism (in the case of vanishing four-current density) is usually based on the Lagrangian

$$L' = -\tfrac{1}{4} F_{\mu\nu} G^{\mu\nu}, \tag{1}$$

where $F_{\mu\nu}$ and $G_{\mu\nu}$ are Minkowski's antisymmetric field-tensors defined by

$$\mathfrak{E} \overset{\text{def}}{=} \{F_{10}, F_{20}, F_{30}\}, \qquad \mathfrak{B} \overset{\text{def}}{=} \{F_{23}, F_{31}, F_{12}\},$$

and

$$\mathfrak{D} \overset{\text{def}}{=} \{G_{10}, G_{20}, G_{30}\}, \qquad \mathfrak{H} \overset{\text{def}}{=} \{G_{23}, G_{31}, G_{12}\}$$

respectively, $\mathfrak{E}$ and $\mathfrak{H}$ being the field vector of the electric and magnetic field, $\mathfrak{D}$ the electric displacement and $\mathfrak{B}$ the magnetic induction *). The tensors $F_{\mu\nu}$ and $G_{\mu\nu}$ are connected by the material equations

$$G_{\mu\nu} v^\nu = \varepsilon F_{\mu\nu} v^\nu,$$

$$G_{\mu\nu} v_\lambda + G_{\nu\lambda} v_\mu + G_{\lambda\mu} v_\nu = 1/\mu \{F_{\mu\nu} v_\lambda + F_{\nu\lambda} v_\mu + F_{\lambda\mu} v_\nu\},$$

where v^μ is the four-velocity of the ponderable matter fulfilling the condition of normalisation:

$$v^\mu v_\mu = 1. \tag{2}$$

Furthermore, the dielectric constant and the magnetic permeability of the ponderable matter are represented by ε and μ respectively. The explicit dependence of $G_{\mu\nu}$ on $F_{\mu\nu}$, ε and μ respectively can be given as follows:

---

*) Our space is pseudo-Euclidean with the co-ordinates $x^0 = ct \equiv t$ (the velocity of light c to be regarded as unity), $x^1 = a$, $x^2 = y$, $x^3 = z$ and with the metrical ground tensor

$$g_{00} = -g_{11} = -g_{22} = -g_{33} = +1, \quad g_{\mu\nu} = 0 \quad \mu \neq \nu; \quad g = \det g_{\mu\nu}.$$

(3) $$G_{\mu\nu}=1/\mu\{F_{\mu\nu}+(\varepsilon\mu-1)[F_\mu v_\nu-F_\nu v_\mu]\}$$

with

$$F_\mu \overset{\text{def}}{=} F_{\mu\lambda}v^\lambda.$$

Based on the supposition that $F_{\mu\nu}$ is the curl of the four-potential $\Phi_\mu=\{-\varphi, \mathfrak{A}\}$ (φ being the scalar and $\mathfrak{A}$ the vector potential respectively) i. e.

$$F_{\mu\nu}=\partial_\mu\Phi_\nu-\partial_\nu\Phi_\mu.$$

Minkowski's equations of the field may usually be regarded as the Euler-Lagrange equations of the variational principle

(4) $$\delta\int\mathfrak{L}d^4\lambda=0, \qquad \mathfrak{L}=L\sqrt{|g|},$$

and can be written as follows:

(5) $$\partial_\nu G^{\mu\nu}=0.$$

The three-dimensional form of the Lagrangian L' —

$$L'=\tfrac{1}{2}\{\mathfrak{H}\mathfrak{B}-\mathfrak{D}\mathfrak{E}\}$$

shows clearly that L' represents an immediate generalisation of the Lagrangian in vacuum:

$$L_0=\tfrac{1}{2}\{\mathfrak{H}^2-\mathfrak{E}^2\}.$$

Undoubtedly, this formal generalisation of the Lagrangian for ponderable matter is a very natural one; from the physical point of view, however, the justification of this generalisation seems inadequate.

A more suitable elaboration of the theory can be suggested as follows:

The electromagnetic field represented by the field tensor $F_{\mu\nu}$ is in dielectrica a non-closed system since the ponderable matter of the dielectrica is polarised by the field. The polarisation of the dielectricum can be represented by the antisymmetric tensor $M_{\mu\nu}$ having the components:

$$\mathfrak{P}=\{M_{01}, M_{02}, M_{03}\}, \qquad \mathfrak{M}=\{M_{23}, M_{31}, M_{12}\},$$

where $\mathfrak{P}$ and $\mathfrak{M}$ are the electric and magnetic polarisations of the ponderable matter. This means that the Lagrangian of the field is made up of the Lagrangian of the field and of the interaction between the field and the current of polarisation respectively (in its general covariant form):

(6) $$L=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}\Phi_\mu\Delta_\nu M^{\mu\nu}\equiv-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}\frac{1}{\sqrt{|g|}}\Phi_\mu\partial_\nu\{\sqrt{|g|}\,M^{\mu\nu}\}.$$

Taking into account that the tensor $M_{\mu\nu}$ is usually defined by

$$G_{\mu\nu}=F_{\mu\nu}-M_{\mu\nu},$$

or explicitly by

$$M_{\mu\nu}=\left(1-\frac{1}{\mu}\right)F_{\mu\nu}-\frac{\varepsilon\mu-1}{\mu}[F_\mu v_\nu - F_\nu v_\mu], \tag{7}$$

it can be immediately seen that the difference of L' and L is a total divergence because of which the variational principle (4) based on the Lagrangian (6) also yields the field equations (5).

2. The following consideration will be based on Lagrangian (7), and in the greater part of this section we assume that our space is a Riemannian one, the structure of which is determined by the metrical ground tensor $\gamma_{\mu\nu}$.

The fact that the integral of action of the field

$$J=\int_\Omega \mathfrak{L}d^4x$$

is an integral invariant is well known. This means that, on changing the co-ordinates x_μ by the infinitesimal transformation [10]

$$x^{\mu'}=x^\mu+\theta\xi^\mu(x), \tag{8}$$

(where ξ^μ is a four-vector, the components of which have to vanish at the boundary of the region of integration Ω and θ represents an infinitesimal parameter), the total variation of J, i. e.

$$\delta J=J'(x')-J(x),$$

vanishes identically. By well known methods it can be immediately proved, that the vanishing of δJ can be written in the following form [3]:

$$\delta J=\int_\Omega \delta^*\mathfrak{L}d^4x=0, \tag{9}$$

where

$$\delta^*\mathfrak{L}=\mathfrak{L}'(x)-\mathfrak{L}(x)$$

is the local variation of $\mathfrak{L}$.

Considering that $\mathfrak{L}$ depends on $\gamma^{\mu\nu}$ and on its first partial derivative ($\gamma^{\mu\nu}(\alpha)=\partial_\alpha\gamma^{\mu\nu}$); on Φ_μ, on its first and second covariant derivative ($\Phi_{\mu|\nu}=\nabla_\nu\Phi_\mu$, $\Phi_{\mu|\nu\lambda}=\nabla_\lambda\Phi_{\mu|\nu}$); on the scalars $\varepsilon=\varepsilon(x)$ and $1/\mu=1/\mu(x)$ respectively, and on their first derivatives ($\varepsilon_{(\alpha)}=\partial_\alpha\varepsilon$; $(1/\mu)_\alpha=\partial_\alpha(1/\mu)$; and finally on $v^\mu=v^\mu(x,\gamma_{\mu\nu}(x))$ and on its first covariant derivative, ($v^\mu{}_{,\alpha}=\Delta_\alpha v^\mu$) — equation (9) can be written as follows:

$$\int_\Omega\Big\{\frac{\partial\mathfrak{L}}{\partial\gamma^{\mu\nu}}\delta^*\gamma^{\mu\nu}+\frac{\partial\mathfrak{L}}{\partial\gamma^{\mu\nu}_{..(\alpha)}}\delta^*\gamma^{\mu\nu}_{..(\alpha)}+\frac{\partial\mathfrak{L}}{\partial\Phi_\mu}\delta^*\Phi_\mu+\frac{\partial\mathfrak{L}}{\partial\Phi_{\mu,\nu}}\delta^*\Phi_{\mu,\nu}+\frac{\partial\mathfrak{L}}{\partial\Phi_{\mu\,\nu\lambda}}\delta^*\Phi_{\mu\,\nu\lambda}+$$
$$+\frac{\partial\mathfrak{L}}{\partial\varepsilon}\delta^*\varepsilon+\frac{\partial\mathfrak{L}}{\partial\varepsilon_{(\alpha)}}\delta^*\varepsilon_{(\alpha)}+\frac{\partial\mathfrak{L}}{\partial(1/\mu)}\delta^*\left(\frac{1}{\mu}\right)+\frac{\partial\mathfrak{L}}{\partial(1/\mu)_{(\alpha)}}\delta^*\left(\frac{1}{\mu}\right)_{(\alpha)}+$$
$$+\frac{\partial\mathfrak{L}}{\partial v^\mu}\delta^*v^\mu+\frac{\partial\mathfrak{L}}{\partial v^\mu_{\ ,\alpha}}\delta^*v^\mu_{\ ,\alpha}\Big\}d^4x=0.$$

By partial integration we have [5]:

$$\int_{\Omega}\left\{-\frac{1}{2}T'_{\mu\nu}\delta^*\gamma^{\mu\nu}+\left[\frac{\partial L}{\partial\Phi_\mu}-\nabla_\nu\left(\frac{\partial L}{\partial\Phi_{\mu|\nu}}-\nabla_\lambda\frac{\partial L}{\partial\Phi^{\mu|\nu\lambda}}\right)\right]\delta^*\Phi_\mu+\left[\frac{\partial L}{\partial\varepsilon}-\partial_\alpha\frac{\partial L}{\partial\varepsilon_{(\alpha)}}\right]\delta^*\varepsilon+\right.$$

$$\left.+\left[\frac{\partial L}{\partial(1/\mu)}-\partial_\alpha\frac{\partial L}{\partial(1/\mu)_{(\alpha)}}\right]\delta^*\left(\frac{1}{\mu}\right)+\left[\frac{\partial L}{\partial v^\mu}-\nabla_\alpha\frac{\partial L}{\partial v^\mu_{\cdot|\alpha}}\right]\delta'^*v^\mu\right\}d^4x=0,$$

where

$$T'_{\mu\nu}=-\frac{1}{\sqrt{|\gamma|}}\left\{\left(\frac{\partial\mathfrak{L}}{\partial\gamma^{\mu\nu}}+\frac{\partial\mathfrak{L}}{\partial\gamma^{\nu\mu}}\right)-\partial_\alpha\left(\frac{\partial\mathfrak{L}}{\partial\gamma^{\mu\nu}_{\cdot\cdot(\alpha)}}+\frac{\partial\mathfrak{L}}{\partial\gamma^{\nu\mu}_{\cdot\cdot(\alpha)}}\right)\right\}\equiv\gamma_{\mu\nu}L-\left(\frac{\partial L}{\partial\gamma^{\mu\nu}}+\frac{\partial L}{\partial\gamma^{\nu\mu}}\right)+$$

$$+\partial_\alpha\left(\frac{\partial L}{\partial\gamma^{\mu\nu}_{\cdot\cdot(\alpha)}}+\frac{\partial L}{\partial\gamma^{\mu\nu}_{\cdot\cdot(\alpha)}}\right)+\{\alpha^\tau\tau\}\left(\frac{\partial L}{\partial\gamma^{\mu\nu}_{\cdot\cdot(\alpha)}}+\frac{\partial L}{\partial\gamma^{\nu\mu}_{\cdot\cdot(\alpha)}}\right)$$

represents the Lagrangian derivative of $\mathfrak{L}$ and the factor of $\delta^*\Phi_\mu$ is precisely identical with the left side of the field equation (5); hence it vanishes. Finally, in the last term,

$$\delta'^*v^\mu\overset{\text{def}}{=}\delta^*v^\mu-\frac{1}{2}\left(\frac{\partial v^\mu}{\partial\gamma^{\varrho\sigma}}+\frac{\partial v^\mu}{\partial\gamma^{\sigma\varrho}}\right)\delta^*\gamma^{\varrho\sigma}$$

is introduced, instead of δ^*v^μ, in view of the fact that $T'_{\mu\nu}$ already contains the variation of v^μ originating from $\gamma^{\mu\nu}$.

Taking the explicit form of $\delta^*\gamma^{\mu\nu}$, $\delta^*\varepsilon$, $\delta^*(1/\mu)$ and δ'^*v^μ respectively into account [10],

$$\int_{\Omega}\theta\left\{\frac{1}{2}\mathfrak{T}_{\mu\nu}\partial\gamma^{\mu\nu}+\partial_\mu\mathfrak{T}_\lambda{}^\mu-\sqrt{|\gamma|}\left[\frac{\partial L}{\partial\varepsilon}-\partial_\alpha\frac{\partial L}{\partial\varepsilon_{(\alpha)}}\right]\partial_\lambda\varepsilon-\sqrt{|\gamma|}\left[\frac{\partial L}{\partial(1/\mu)}-\partial_\alpha\frac{\partial L}{\partial(1/\mu)}\right]\partial_\lambda\left(\frac{1}{\mu}\right)-\right.$$

$$\left.-\partial_\varrho\left[\sqrt{|\gamma|}\left(\frac{\partial L}{\partial v^\lambda}-\nabla_\alpha\frac{\partial L}{\partial v^\lambda_{|\alpha}}\right)v^\varrho-\sqrt{|\gamma|}\left(\frac{\partial L}{\partial v^\mu}-\nabla_\alpha\frac{\partial L}{\partial v^\mu_{\cdot|\alpha}}\right)v^\mu v_\lambda v^\varrho\right]\right\}\xi^\lambda d^4x=0$$

is obtained. However, this is an identity for arbitrary ξ^λ, and for arbitrary choice of the integration region; therefore, in the pseudo-Euclidean space ($\gamma_{\mu\nu}\to g_{\mu\nu}$), we have the following identity:

$$\partial_\mu T_\lambda{}^\mu-\left[\frac{\partial L}{\partial\varepsilon}-\partial_\alpha\frac{\partial L}{\partial\varepsilon_{(\alpha)}}\right]\partial_\lambda\varepsilon-\left[\frac{\partial L}{\partial(1/\mu)}-\partial_\alpha\frac{\partial L}{\partial(1/\mu)_{(\alpha)}}\right]\partial_\lambda\left(\frac{1}{\mu}\right)-$$

$$-\partial_\varrho\left\{\left[\frac{\partial L}{\partial v^\lambda}-\partial_\alpha\frac{\partial L}{\partial v^\lambda_{\cdot(\alpha)}}\right]v^\varrho-\left[\frac{\partial L}{\partial v^\mu}-\partial_\alpha\frac{\partial L}{\partial v^\mu_{\cdot(\alpha)}}\right]v^\mu v_\lambda v^\varrho\right\}=0,\tag{10}$$

where

$$T_\lambda{}^\mu\overset{\text{def}}{=}\{T'^\mu_\lambda\}\gamma_{\mu\nu}=g_{\mu\nu}.$$

These equations are the required identities which give the laws of conservation for the electromagnetic field in ponderable matter.

3. Taking into account the explicit form of our Lagrangian (6), we obtain for $T_{\nu\mu}$ Abraham's energy-momentum tensor:

$$\underset{(A)}{T_{\mu\nu}} = \underset{(M)}{T_{\mu\nu}} + T^{*}_{\mu\nu}, \tag{10}$$

where

$$\underset{(M)}{T_{\mu\nu}} \overset{\text{def}}{=} F_{\mu\varrho} G_{\nu\cdot}^{\ \varrho} - \tfrac{1}{4} g_{\mu\nu} F_{\varrho\sigma} G^{\varrho\sigma} \tag{11}$$

is the so-called Minkowski energy-momentum tensor, and

$$T^{*}_{\mu\nu} \overset{\text{def}}{=} \frac{\varepsilon\mu - 1}{\mu} [F_{\mu\varrho} F^{\varrho} v_{\nu} + F_{\varrho} F^{\varrho} v_{\mu} v_{\nu}]. \tag{12}$$

Since, based on elementary calculation, the laws of conservation (9) can be written either in the form

$$\partial_{\varrho} \underset{(A)}{T_{\lambda\cdot}^{\ \varrho}} + \frac{1}{2} F_{\varrho} F^{\varrho} \partial_{\lambda} \varepsilon + \frac{1}{4} \left[F_{\varrho\sigma} F^{\varrho\sigma} - F_{\varrho} F^{\varrho} \right] \partial^{\lambda} \left(\frac{1}{\mu} \right) - \partial_{\varrho} \left[\frac{\varepsilon\mu - 1}{\mu} (F_{\lambda\alpha} F^{\alpha} v^{\varrho} + \right. \tag{13}$$

$$\left. + F_{\alpha} F^{\alpha} v^{\lambda} v^{\varrho}) \right] = 0,$$

or, using (10), as follows:

$$\partial_{\varrho} \underset{(M)}{T_{\lambda\cdot}^{\ \varrho}} + \frac{1}{2} F_{\varrho} F^{\varrho} \partial_{\lambda} \varepsilon + \frac{1}{4} [F_{\varrho\sigma} F^{\varrho\sigma} - F_{\varrho} F^{\varrho}] \partial_{\lambda} \left(\frac{1}{\mu} \right) = 0, \tag{14}$$

it can be proved by direct calculation that both equations are identically fulfilled.

This means, however, that both rival energy-momentum tensors of the electromagnetic field in ponderable matter — Minkowski's $\underset{(M)}{T_{\nu\mu}}$ as well as Abraham's $\underset{(A)}{T_{\mu\nu}}$ — fulfil the true laws of conservation, and hence, in opposition to the arguments of previous authors [1], [4], [7], [9], the laws of conservation are not suitable for settling the dispute concerning the two tensors of energy and momentum.

For the decision there still remains Laue's criterion of radiation [6] which — it is well known — decides in favour of Minkowski's tensor. This decision has also been confirmed more recently by Rubinowicz's interesting arguments [11].

It is, of course, true that Minkowski's tensor does not fulfil Planck's criterion, although that is fulfilled automatically by Abraham's. However, this is only due to the fact that the electromagnetic field in ponderable matter is a non-closed system [12]. It is well known that owing to Beck's method [2], a so-called radiation tensor can be introduced, which, considering that the closed system is made up by the matter and the field, represents the total energy-momentum tensor of the closed system, fulfilling both Planck's and Laue's criterions.

Recently Marx and Györgyi [7] as well as Marx and Nagy [8] have given a method for the construction of the total energy-momentum tensor of matter, and of a field based on Abraham's tensor, which is very similar to Beck's, and have given the tensor of radiation in the following explicit form:

$$S_{\mu\nu} = \underset{(A)}{T_{\mu\nu}} - \frac{\varepsilon\mu - 1}{\varepsilon\mu}(g_{\mu\varrho} - v_\mu v_\varrho)(g_{\nu\sigma} - v_\nu v_\sigma)\underset{(A)}{T^{\varrho\sigma}}.$$

However, it can be proved by direct calculation that

$$(g_{\mu\varrho} - v_\mu v_\varrho)(g_{\nu\varrho} - v_\nu v_\sigma) T^{*\varrho\sigma} \equiv 0,$$

and therefore $S_{\mu\nu}$ can be written as

$$S_{\mu\nu} = \underset{(M)}{T_{\mu\nu}} + T^*_{\mu\nu} - \frac{\varepsilon\mu - 1}{\varepsilon\mu}(g_{\mu\varrho} - v_\mu v_\varrho)(g_{\nu\sigma} - v_\nu v_\sigma)\underset{(M)}{T^{\varrho\sigma}}.$$

Since, $T^*_{\mu\nu}$ can be constructed on the basis of the postulate that $\underset{(M)}{T_{\mu\nu}} + T^*_{\mu\nu}$ must fulfil Planck's criterion, it is obvious that, on the one hand, by means of the method of Marx and his co-workers the total tensor of matter and field can be deduced from Minkowski's tensor, but only in two steps, and that, on the other hand, it can be directly deduced by Beck's method, too. So far it has not been proved that the above two energy-momentum tensors of the total system are identical; it does, however, seem obvious that they are equivalent.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 7, 1956

THEORETICAL PHYSICS

Determination of the Valence Band in Metallic Sodium by the Parzen Variational Method

by

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Presented by L. INFELD on May 3, 1956

The variational method proposed by Parzen [1] and applied by Wainwright and Parzen [2] to metallic lithium, was used to determine the valence band of metallic sodium.

To describe the electrons in the crystal, Wannier functions were used. The actual function must minimize the functional (H is the Hamiltonian of the crystal as a whole)

$$I_0=\int U_v^*(\boldsymbol{r})\,H\,U_v(\boldsymbol{r})\,d\boldsymbol{r}, \tag{1}$$

with the side conditions

$$\int U_v^*(\boldsymbol{r})\,U_v(\boldsymbol{r}+\boldsymbol{r}_n)\,d\boldsymbol{r}=\delta_{n,0}, \tag{2}$$

$$\int U_v^*(\boldsymbol{r})\,U_q(\boldsymbol{r}+\boldsymbol{r}_n)\,d\boldsymbol{r}=0. \qquad n=0,\pm1,\pm2,\ldots \tag{3}$$

Here, the subscript v denotes the valence function, while q denotes the functions of the lower states, that is for sodium the states $1s$, $2s$, $2p$.

Using, further, the notation analogic to that of paper [2] we assume the trial function in the form

$$U_v(\boldsymbol{r})=W(\boldsymbol{r})-\sum_j\sum_q C_{q,j}U_q(\boldsymbol{r}-\boldsymbol{r}_j). \tag{4}$$

The coefficients $C_{q,j}$ are determined from the orthogonality conditions (3). We assume further the W function in the form of a Fourier development:

$$W(\boldsymbol{r})=(N\Omega)^{-1/2}\sum_{\boldsymbol{k}}\sum_n b(\boldsymbol{k}+\boldsymbol{k}_n)\exp\left[i(\boldsymbol{k}+\boldsymbol{k}_n)\boldsymbol{r}\right]. \tag{5}$$

Here, $\boldsymbol{k}_n$ denote the vectors of the reciprocal lattice, while $\boldsymbol{k}$ denotes the reduced vector. N is the total number of atoms in the crystal, Ω_0 is the volume of the unit cell and $\Omega=N\Omega_0$.

The trial functions are now the $b(\boldsymbol{k}+\boldsymbol{k}_n)$. The plane wave development (5) is advantageous, for we know something about the behaviour of $b(\boldsymbol{k}+\boldsymbol{k}_n)$ for entirely free and almost free electrons. This leads us to a proper choice of the trial functions. We take

$$b(\boldsymbol{k})=a_{0,0}+a_{0,1}\sum_l \exp(i\boldsymbol{k}\boldsymbol{r}_l), \tag{6}$$

$$b(\boldsymbol{k}+\boldsymbol{k}_{n'})=a_{1,0}.$$

$a_{n,m}$ are now parameters of our variational problem. We retain only three of them. $\boldsymbol{r}_l$ denote here the lattice vectors from the central atom to its 8 nearest neighbours and $\boldsymbol{k}_{n'}$ – the vectors from the central point of the reduced zone to the 12 nearest neighbours in the reciprocal lattice.

Introducing (5), (6) into (4) and substituting (4) into (1) and (2) we obtain:

$$\begin{aligned} I_0=&\, a_{0,0}^2(\mathcal{E}_0+V_0)+8a_{0,1}^2(\mathcal{E}+V_0)+12a_{1,0}^2(\mathcal{E}_0+k_1^2+V)\\ &+16a_{0,0}a_{0,1}\mathcal{E}_1+24a_{0,0}a_{1,0}V_1\\ &-E_{1s}(C_{1s,0}^2+8C_{1s,1}^2+6C_{1s,2}^2)-E_{2s}(C_{2s,0}^2+8C_{2s,1}^2+6C_{2s,2}^2)\\ &-E_{2p}(24C_{2p,1}^2+6C_{2p,2}^2), \end{aligned} \tag{7}$$

$$\begin{aligned} a_{0,0}^2+8a_{0,1}^2+12a_{1,0}^2=&\,1+(C_{1s,0}^2+8C_{1s,2}^2+6C_{1s,2}^2)\\ &+(C_{2s,0}^2+8C_{2s,1}^2+6C_{2s,2}^2+(24C_{2p,1}^2+6C_{2p,2}^2), \end{aligned} \tag{7a}$$

$$a_{0,0}a_{0,1}=C_{1s,0}C_{1s,1}+3C_{1s,1}C_{1s,2}+C_{2s,0}C_{2s,1}+3C_{2s,1}C_{2s,2}+3C_{2p,1}C_{2p2}. \tag{7b}$$

Here,

$$\mathcal{E}=\mathcal{E}_0+3\mathcal{E}_2+3\mathcal{E}_3+\mathcal{E}_5,$$
$$V=V_0+4V_1+2V_2+4V_3+V_4.$$

$\mathcal{E}_j$ and V_j originate respectively from the development into Fourier series of k^2 and of the total crystal potential. E_{1s}, E_{2s}, E_{2p} are eigenvalues of lower states of energy. They have been taken from [3]. From the same paper, the approximate analytical expressions for $U_q(\boldsymbol{r})$ functions have been adopted. Having them, we calculate, by a straightforward integration over the dodecahedron surrounding the first Brillouin zone, the orthogonality coefficients:

$$\begin{aligned} C_{q,l}=&\int W(\boldsymbol{r})\,U_q(\boldsymbol{r}-\boldsymbol{r}_l)\,d\boldsymbol{r}\\ =&\frac{\Omega_0^{1/2}}{(2\pi)^3}\sum_n\sum_j a_{n,j}\int d\boldsymbol{k}\Big\{\exp[i\boldsymbol{k}(\boldsymbol{r}_j+\boldsymbol{r}_l)]\int d\boldsymbol{r}\exp[i(\boldsymbol{k}+\boldsymbol{k}_n)\boldsymbol{r}]\,U_q(\boldsymbol{r})\Big\}\\ =&\frac{\Omega_0^{1/2}}{(2\pi)^3}\sum_n\sum_j a_{n,j}\int d\boldsymbol{k}\exp[i\boldsymbol{k}(\boldsymbol{r}_j+\boldsymbol{r}_l)]f_q(\boldsymbol{k}+\boldsymbol{k}_n). \end{aligned} \tag{8}$$

$a_{n,j}$ are here $a_{0,0}$, $a_{0,1}$, $a_{1,0}$; $f_q(\boldsymbol{k}+\boldsymbol{k}_n)$ are the Fourier transforms of the

atomic functions $U_q(\boldsymbol{r})$. To integrate them over the dodecahedron it was necessary to develop them into the powers of a small quantity $|\boldsymbol{k}+\boldsymbol{k}_n|^2/\gamma_q^2$, where γ_q is the constant in the exponent of a hydrogen-like atomic function $U_q(\boldsymbol{r})$ of the isolated atom.

The lattice constant, $a=8.008$ Bohr units, was adopted. The Fourier coefficients of the crystal potential have been taken from the table published by Weinstein [4]. In rydbergs:

$$V_0=-0.884, \quad V_1=-0.184, \quad V_2=-0.131,$$
$$V_3=-0.109, \quad V_4=-0.099.$$

From the condition $I_0=0$ with (7a, b) we get a set of algebraic equations, which we solve numerically. We obtain the following values for our variational parameters:

$$a_{0,0}=1.0136, \quad a_{0,1}=0.00036, \quad a_{1,0}=-0.0166.$$

This set has been chosen among the other possible solutions as the only reasonable one from the physical point of view.

The energy in the valence band is given by

$$(9) \qquad E(\boldsymbol{k})=\sum_n I_n \exp(i\boldsymbol{k}\boldsymbol{r}_n)=\sum_n \exp(i\boldsymbol{k}\boldsymbol{r}_n)\int U_v^*(\boldsymbol{r}+\boldsymbol{r}_n)HU_v(\boldsymbol{r})d\boldsymbol{r}.$$

Substituting here for the variational parameters the numerical values obtained, the final expression for energy becomes:

$$\begin{aligned}(10) \qquad E(\boldsymbol{k})=&A+Bk^2+\cos\xi\cos\eta\cos\zeta(C+Dk^2)+F\cos^2\xi\cos^2\eta\cos^2\zeta\\&+G(\cos^2\xi+\cos^2\eta+\cos^2\zeta)+H(\cos^4\xi+\cos^4\eta+\cos^4\zeta)\\&+J(\cos^2\xi\cos^2\eta+\cos^2\xi\cos^2\zeta+\cos^2\eta\cos^2\zeta)\\&+K\cos\xi\cos\eta\cos\zeta(\cos^2\xi+\cos^2\eta+\cos^2\zeta).\end{aligned}$$

Here

$$\xi=\frac{a}{2}k_x, \quad \eta=\frac{a}{2}k_y, \quad \zeta=\frac{a}{2}k_z.$$

Energy $E(\boldsymbol{k})$ being measured in rydbergs, the constants have the values:

$$A=-0.6424, \quad B=1.0307, \quad C=0.0562, \quad D=0.0059, \quad F=-0.0530,$$
$$G=0.0071, \quad H=-0.0027, \quad J=0.0179, \quad K=-0.0136$$

Comparison of our results with those obtained in [5] was made in the following points:

$k=\frac{\pi}{a}$ (0,0,0) the midpoint of the zone,

(1,1,0) the midpoint of the dodecahedron face,

(1,1,1) the nearer vertex of the dodecahedron,

(0,0,2) the further vertex of the dodecahedron.

On the right hand of the table, the energies are given in electron-volts, assuming the energy value in the centre of the zone to be equal to 0.

TABLE I

$a/\pi\, k$	$E(k)$ in ry = 13,605 eV				$E(k)$ in eV ($E(0)=0$)		
	(0, 0, 0)	(1, 1, 0)	(1, 1, 1)	(0, 0, 2)	(1, 1, 0)	(1, 1, 1)	(0, 0, 2)
Our results	−0.613	−0.321	−0.167	−0.0130	3.975	6.074	8.162
H.-J. results	−0.608	−0.315	−0.110	−0.0135	3.94	6.72	8.03

We intended to admit one more parameter $a_{1,1}$ in the trial function, but it was found to be so small that it could have no appreciable influence on the energy value with our accuracy of computation. We also admitted $C_{q,3}$ to calculate $C_{q,l}$ more accurately. But this made only an imperceptible change in $a_{0,0}$, $a_{0,1}$, $a_{1,0}$. The quantities, which must be known with great exactness to obtain reasonable energy values, are the potential coefficients V_j, in particular V_0.

I should like to express my thanks to Professor L. Infeld for sustained interest and encouragement and to Professors J. Werle and M. Suffczyński for many profitable discussions.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 7, 1956

Delay-Effect in Ferroelectric Titanates Below the Curie Point

by

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Presented by A. SOŁTAN on May 5, 1956

1. Introduction

After the application, as after the removal, of an external direct-current field, the permittivity of polycrystalline barium and barium-strontium metatitanates does not take on a definite constant value immediately, but does so only after a not inconsiderable delay. This delay-effect had previously been investigated both below and above the Curie point by bridge methods which, however, enable the measurement of the time dependence of permittivity to begin only some seconds after the change of the field [2], [4]. It is desirable to know more about this effect for considerably shorter times after the change of the field. Moreover, the authors suspected the existence of such an effect in single crystals of barium metatitanate. Hence observation was made of short-time delay-effects in barium and barium-strontium metatitanate ceramic, as well as in $BaTiO_3$ single crystals, below the Curie point.

2. Experimental

In order to observe the short-time effects, the oscillographic method shown in Fig. 1 was employed. The sample measured represents a part of a high frequency resonant circuit of one of two receivers fed by an oscillator with a frequency of

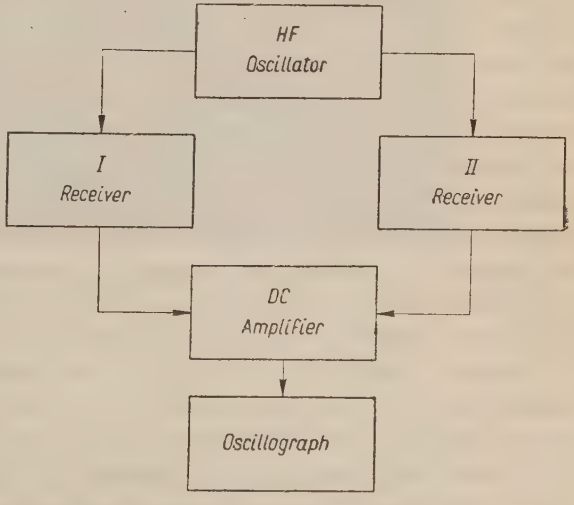


Fig. 1. Schema of the measuring apparatus

3.2 Mc. per sec. The measuring HF voltage on the samples reached 1 V. The detected differential output signal, proportional to the permittivity changes of the sample, was applied to the DC amplifier of an oscillograph. Releasing the inhibited time base synchronously with the change of the field, the single oscillograms for ceramic samples and $BaTiO_3$ crystals under test were obtained. Oscillograms for samples investigated were made at room temperature both on the left and the right slope of the resonance curve. Convenient sweep frequencies were from 2 cps. to 150 cps. The DC field applied to the samples was up to 4 kV per cm., having the form of a rectangular impulse longer than 1 min. Inhibiting the time base, long-time observations also were made and showed a satisfactory measure of agreement with those obtained by the bridge methods referred to above. Further tests of the measuring method with an air or mica condenser did not indicate any changes in capacity either following the appearance or disappearance of the field.

Polycrystalline material of various composition was produced by the known procedure [1]. The tablets covered with silver electrodes on both faces were about 3 mm. thick. They were placed in a paraffin oil bath to eliminate any temperature change influence on the delay-effect investigated.

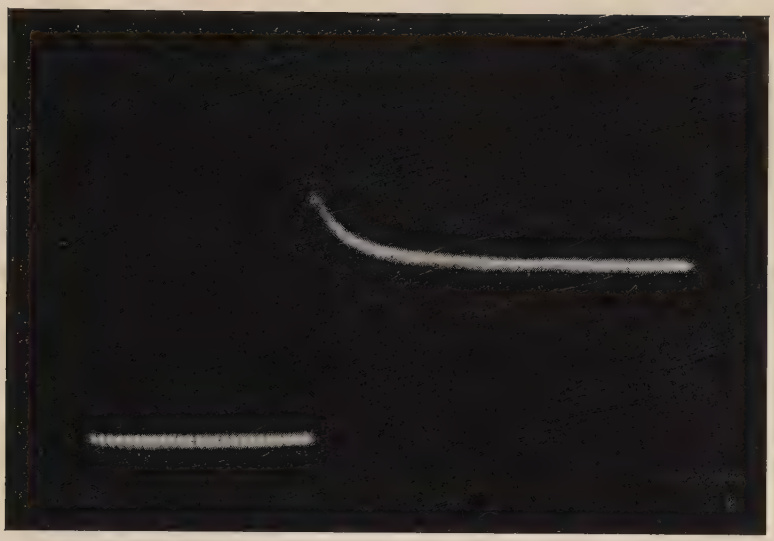
The $BaTiO_3$ single crystals grown by W. Pajewski in the Warsaw Industrial Institute of Telecommunication were about 0.15 mm. thick. The electrodes on these crystals were made by Z. Bochyński in our Laboratory.

3. Results

The material used was $BaTiO_3$ ceramic and crystals and barium-strontium metatitanate ceramic with a composition of $77\%\,BaTiO_3 + 23\%\,SrTiO_3$.

It was found that the application or removal of an external field involved a sudden *increase* of permittivity in a ferroelectric up to a certain maximum, followed by a slow decrease. The oscillograms in Figs. 2a and 2b show the delay-effect in barium metatitanate ceramic after the application and removal of the field, respectively. The time in which the maximum value of permittivity was reached is not, as far as the authors were able to ascertain by the sweep frequencies employed, greater than that necessary for the increase or decrease of the field. However, on removal of the field, the decrease of permittivity proceeded at a slower rate than after the application of the DC field. The maximum value of ε in the delay-effect increases with the intensity of the field.

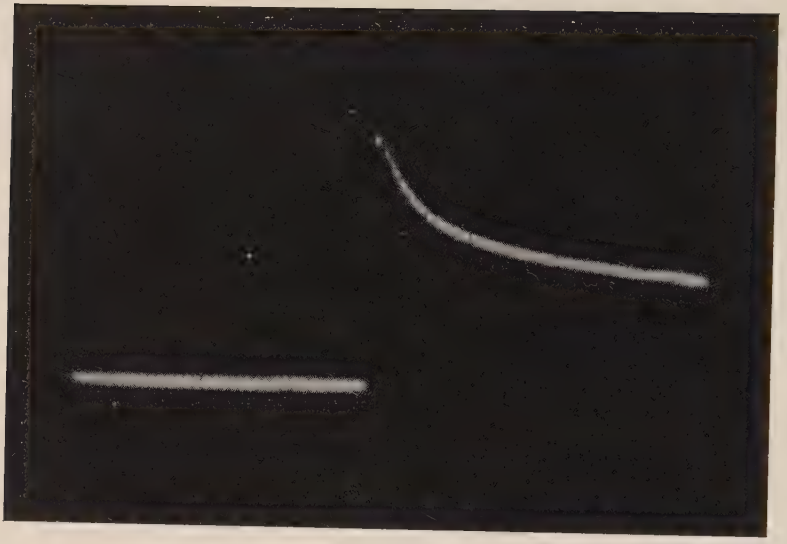
Similar results were obtained for barium-strontium metatitanate ceramic and, what is important, also for $BaTiO_3$ single crystals. Oscillograms in Figs. 3a and 3b show the delay-effect in a $BaTiO_3$ single crystal. It is clearly apparent that the character of the effect is the same as for ceramic samples. For weak fields, the maximum value of permittivity ε_m



(a)

(b)

Fig. 2. Delay-effect in $BaTiO_3$ ceramic after application (a) and removal (b) of the field (333 V per cm.), by sweep frequency 21 cps.



(a)

(b)

Fig. 3. Delay-effect in $BaTiO_3$ single crystal after the application (a) and removal (b) of the field (4 kV per cm.), by sweep frequency 30 cps.

s almost proportional to the field strength. The relative magnitude, $\varepsilon_m/\varepsilon$ at 800 V per cm. amounted to about 2% in both ceramic and the single crystal.

Further observations showed that after a longer period of time, the permittivity decreased permanently and tended to remain constant, approaching a level below the initial value by application, or a level equal to the initial value by removal of the field. The difference between the initial value and that when applying the field, increases with the latter and also depends on the thermal and electric history of the sample investigated [4].

4. Discussion

The appearance of the delay-effect observed is of importance for certain radio-devices containing ferroelectric elements. Our results show that the delay-effect appears even in $BaTiO_3$ single crystals. Thus it seems that this effect must arise as a result of imperfections in the crystal lattice. As suggested by A. Piekara [3], these imperfections may cause an increase of the potential barrier between two equilibrium positions of the Ti^{+4} ion in the oxygen octahedron of the perovskite structure. This involves a considerable increase in the relaxation time of ions in the crystals investigated. The relaxation time of the ion can be expressed as [5], [6]

$$\tau \approx 10^{-13} \exp H/kT,$$

where H is the height of the potential barrier. The value of H reported amounts normally to about 0.2 eV, leading to $\tau \approx 10^{-10}$ sec. It is sufficient to admit a value of $H = 0.75$ eV in the distorted lattice to obtain a delay time of the order of seconds for room temperature. It is possible that we have to deal with a number of discrete relaxation times of Ti^{+4} ions, which involve the delay-effect observed.

The authors wish to express their thanks to Professor A. Piekara, under whose direction they worked, for his interest in the present investigation.

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 7, 1956

CHEMISTRY

On Reaction of 1-Nitromethylcyclohexanol and Its Halogen Derivatives with Formaldehyde and Acetaldehyde

by

T. URBAŃSKI, Z. ECKSTEIN and H. WOJNOWSKA

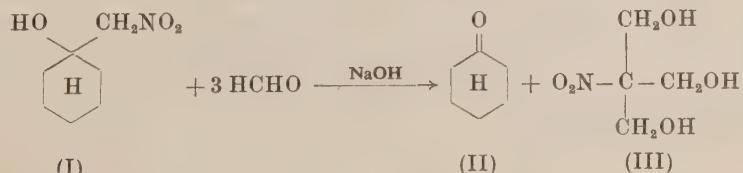
Communicated by T. URBAŃSKI at the meeting of March 19, 1956

It is a well known fact, that certain secondary nitroalcohols, formed from nitroparaffins and homologues of formaldehyde can readily be subjected to a cleavage under action of formaldehyde in alkaline medium. As a result, the aliphatic aldehyde (homologue of formaldehyde) and a primary nitroalcohol, deriving from nitroparaffin and formaldehyde are formed [1], [2]:

$$R_1-\underset{\displaystyle R_2}{\overset{\displaystyle NO_2}{\underset{|}{\overset{|}{C}}}}-\underset{\displaystyle OH}{\underset{|}{CHR}}+HCHO \xrightarrow{[OH^-]} RCHO+R_1-\underset{\displaystyle R_2}{\overset{\displaystyle NO_2}{\underset{|}{\overset{|}{C}}}}-CH_2OH$$

In the case of R_1 and/or $R_2=H$, i. e. in the case of the presence of at least one active hydrogen atom, the mechanism of the reaction is probably such that formaldehyde is added in the first stage, as a result of a kind of aldolisation. Then, the molecule of aldehyde RCHO is split off, and formaldehyde is added in its place.

In the present paper the action of formaldehyde on 1-nitromethylcyclohexanol (I), the known tertiary alcohol, formed from cyclohexanon and nitromethane (according to the literature) in presence of sodium hydroxide in aqueous methanol [3] has been studied. It has now been found, that the compound (I) when reacting with an excess of formaldehyde (3 molecules to 1 molecule or (I)), in the presence of sodium hydroxide at room-temperature yielded tri-(hydroxymethyl)-nitromethane (III), and cyclohexanon (II) was split off:

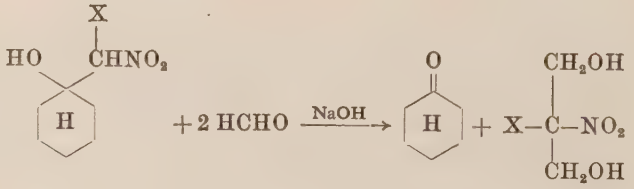


(I) (II) (III)

The products were separated by fractional extraction with ether. In the first instance cyclohexanon was extracted and identified as 2,4-dinitrophenylhydrazone. Next tri-(hydroxymethyl)-nitromethane (III) (the yield c. 79.5% of theoretical) was extracted with ether. The product (III) (m. p. 153-155°) was transformed into 5-nitro-5-hydroxymethyl-2-phenyl-1,3-dioxane, which was found to be identical with the product described in one of the former papers [4].

Two new derivatives of (I) have also been prepared: 1-chloronitromethylcyclohexanol (IVa) and 1-bromonitromethylcyclohexanol (IVb) by acting with chlorine or bromine on the sodium salt of 1-nitromethylcyclohexanol (a method analogous to that used by Schmidt and Wilkendorf [5]). We expected the halogen derivatives of (I) to be more stable than the mother substance (I). This assumption was based on the fact described in one of our earlier papers [2], where the presence of a halogen atom attached to carbon bonded with a nitrogroup was found to increase the stability of the nitroparaffin. It was possible to obtain mixed nitrodiols by reacting with formaldehyde.

It has now been found that in the case of the halogen derivatives of the tertiary alcohol (I), (i. e. of compounds (IV)) no change of reactivity occurred and both halogenetad products reacted with formaldehyde in presence of sodium hydroxide to yield cyclohexanone (II) as well as 2-chloro-2-nitro-1,3-propanediol (Va) and 2-bromo-2-nitro-1,3-propanediol (Vb) respectively:



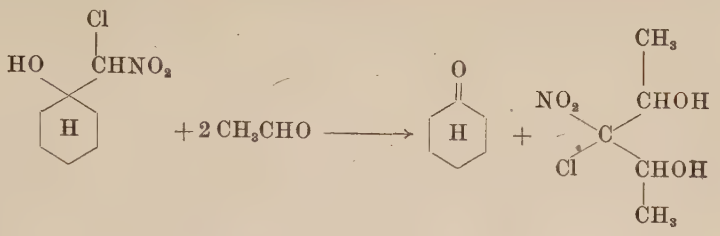
(IV) (II) (V)

(IVa) X = Cl (Va) X = Cl

(IVb) X = Br (Vb) X = Br

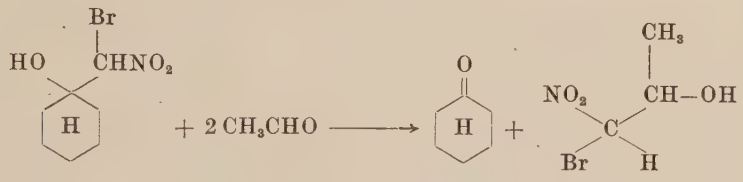
The compounds (Va) and (Vb) are identical with the compounds obtained formerly in another way [6], [7].

The reaction of 1-chloro- and 1-bromonitromethylcyclohexanol [(IVa) and (IVb) respectively] with acetaldehyde in presence of sodium hydroxide has also been examined. A similar trend of the reaction has been found in the case of (IVa), as 3-chloro-3-nitropentane-2,4-diol (VI) and cyclohexanon resulted:



(IVa) (II) (VI)

On the contrary, when 3-bromo-3-nitropentane-2,4-diol (IVb) was used, only a monohydric alcohol, 1-bromo-1-nitro-2-propanol (VII), and cyclohexanone were formed:



(IVb) (II) (VII)

The compounds (VI) and (VII) are known in literature [1], [8].

Experimental details referring to the present paper will be given elsewhere [9].

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 7, 1956

CHEMISTRY

The Seith-Wever Effect Interpreted from the Point of View of Thermodynamics of Irreversible Processes

by

B. BARANOWSKI

Presented by M. ŚMIAŁOWSKI on April 4, 1956

Seith and Wever recently discovered a new effect in electrolytic transference in solid binary alloys; the essence of that effect is independent migration of both metallic components in the electric field [1]. This resulted, in the phase β of Cu_3Al investigated by the authors, in the disappearance of lattice points at the cathode, while new lattice planes developed simultaneously at the anode. This effect resembles the migration of inert indicators in the process of isothermic diffusion, observed by Kirkendall [2]. The independent migration of metallic components taking place in solid state in transport phenomena and repeatedly observed in recent years, has made it possible to introduce partial diffusion coefficients and partial transport numbers as directly measurable quantities [3]. Wever [3] attempted to rest the theory connecting partial diffusion coefficients with partial transport numbers on an earlier theory of Wagner [4]. However, this theory may be subjbect to serious reservations since, in order to explain the simultaneous movement towards the anode of copper and aluminium observed by Seith and Wever [1], metal ions must be assumed to be negatively charged. The difficulty inherent in this theory results from neglecting the effect on the flows of metallic components of electron movement [5] [6]. This kind of reciprocal effect of various flows in transport processes can be assessed quantitatively with the aid of thermodynamics of irreversible processes [7]. The interpretation of the Seith-Wever effect, presented further below, is based on the latter theory.

Thermodynamic forces, flows, phenomenological coefficients

Let us examine a solid binary metal alloy at a constant temperature. Molar flows of metallic components and of electrons, as referred to a fixed

system, may be expressed by the equation [7]:

$$J_k = \sum_{i=1}^{3} L_{ki} X_i \quad (k = 1,2,3), \tag{1}$$

where the indices 1 and 2 refer to metal ions, and index 3 refers to electrons. L_{ik} denotes phenomenological coefficients, X_i – thermodynamic forces expressed in the case discussed by the formulae:

$$X_i = -\operatorname{grad} \mu_i - e_i \operatorname{grad} \varphi, \tag{2}$$

where μ_i denotes the chemical potential, e_i – the molar charge of the given component, and φ – the electric potential.

Equation (1) contains 9 phenomenological coefficients, of which only 6 are independent in agreement with the reciprocal relations:

$$L_{ik} = L_{ki}. \tag{3}$$

Furthermore, assuming the absence of interaction between the flows of metallic components,

$$L_{12} = 0, \tag{4}$$

we reduce the number of independent phenomenological coefficients to five. Equation (4) is in agreement with the independent movement of metallic components discussed in the introduction. Furthermore, we introduce the transfer quantities:

$$\left(\frac{J_3}{J_1}\right)_{x_2 = x_3 = 0} = \frac{L_{13}}{L_{11}} = e_1^*; \qquad \left(\frac{J_3}{J_2}\right)_{x_3 = x_1 = 0} = \frac{L_{23}}{L_{22}} = e_2^*. \tag{5}$$

Values e_1 and e_2^* represent a quantitative expression of the interaction between the flow of electrons and that of metallic components.

Using the definitions of partial transport numbers (t_1 and t_2) and of the specific conductance of the alloy, we express the remaining three phenomenological coefficients as follows:

$$t_1 = \left(\frac{J_1}{J_3}\right)_{\operatorname{grad}\mu_1 = \operatorname{grad}\mu_2 = 0} \approx -\frac{L_{11}(e_1 - e_1^* F)}{L_{33} F}, \tag{6}$$

$$t_2 = \left(\frac{J_2}{J_3}\right)_{\operatorname{grad}\mu_1 = \operatorname{grad}\mu_2 = 0} \approx -\frac{L_{22}(e_2 - e_1^* F)}{L_{33} F}, \tag{7}$$

$$J_{33} \approx L_{33} F \operatorname{grad} \varphi = \frac{\varkappa}{F} \operatorname{grad} \varphi. \tag{8}$$

Relation between partial diffusion coefficients and partial transport numbers

The phenomenological equations quoted above describe the flows of metallic components and of electrons taking place both during the process of isothermic diffusion and during that of migration in the elec-

tric field. Consequently, they may serve as a basis for deriving equations connecting partial diffusion coefficients with partial transport numbers, which was the object of Wever's theory [3]. To do this, we use the definition of the partial diffusion coefficient D_i:

$$\boldsymbol{J}_i = -D_i \operatorname{grad} c_i, \tag{9}$$

where c_i denotes the concentration of the component i in mols/ccm.

Substituting for $\boldsymbol{J}_i$ the expression quoted, and using the formulae for the phenomenological coefficients, we obtain the following equation connecting the partial transport number of the given component with its partial diffusion coefficient:

$$t_i = -\frac{F(e_i - e_i^* F)}{RT\varkappa} c_i D_i \frac{d \ln \gamma_i}{d \ln a_i} \qquad (i = 1,2), \tag{10}$$

where γ_i denotes the molar fraction of the given component, and a_i its activity.

It should be added that formula (10) was obtained after the following additional approximations:

a) the diffusion potential was completely disregarded in the equations for the thermodynamic forces as introduced in equation (9);

b) in the equation for the thermodynamic force referred to electrons, the electron concentration gradient was disregarded.

Discussion

Equation (10) explains the possibility of both metal ions moving in electrolytic transfer towards the cathode, as well as towards the anode. This possibility is explained by accounting for the effect on the direction of transference of the charge of metal ions, as well as of the interaction between the flows of electrons and of metal ions. It follows from the equation quoted, that the direction in which the given component migrates is determined by the value, positive or negative, of the expression $e_i - e_i^* F$. Assuming a positive value for e_i^*, migration towards the anode takes place when the interaction between the flows of electrons and that of metal ions, expressed by the value e_i^*, exceeds the effect of the charge of metal ions on the direction of transference.

The equations quoted demonstrate that, contrary to the assumption of Seith and Wever [1], partial transport numbers and partial diffusion coefficients do not suffice to determine the charges of metal ions. We merely obtain a certain value connected with the charges of metal ions as well as with the interaction between the flows of metallic components and of electrons. This fact makes it superfluous to ascribe negative charges to metal ions moving towards the anode [3], and thus removes the fundamental difficulty of both Wagner's [4] and Wever's [3] theories.

The relation between the charges of aluminium and of copper in the β phase of Cu_3Al, determined by Seith [3] (on the basis of Wever's theory), might be indicative of the values e_i^* in the given alloy being proportionate to the charges of metal ions.

It can be easily proved that equation (10) is reduced to the expression derived by Wever [3], when the interaction between the flows of electrons and of metallic components is neglected -- i. e. when assuming $e_1^* = e_2^* = 0$.

Details and others results of the thermodynamic interpretation of the Seith-Wever effect will be published elsewhere.

INSTITUTE OF PHYSICAL CHEMISTRY, POLISH ACADEMY OF SCIENCES

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BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 7, 1956

GEOLOGY

Preferred Orientation of Pebbles in Recent Gravels of the Dunajec River in the Western Carpathians

by

R. UNRUG

Presented by M. KSIĄŻKIEWICZ on March 30, 1956

The connection of the orientation of pebbles with the direction of the transporting current is widely known and obviously useful in paleogeographic reconstruction work. It consists in the imbrication of pebbles with dips of the largest cross-section planes, directed against the current. The scope of knowledge regarding the dependence of the preferred orientation of pebbles on their size and shape is limited, even though such dependence is admitted beyond doubt. Cailleux [1] mentions that imbrication is more distinct in larger pebbles than in smaller ones. Krumbein [2] has shown the orientation of the longest axes to be more distinct in pebbles with low sphericity.

With a view to correlating the orientation of pebbles with their size and shape, measurements of the orientation of axes have been taken in pebbles belonging to various grade sizes, and various shape classes, these being followed after Zinng's classification [3]. Measurements were taken of pebbles collected from the superficial layer of lag gravel along the banks of the Dunajec River in the Western Carpathians. These gravels are referable to one sedimentary unit, deposited under more or less uniform conditions. Comparability of samples is based on the rule of collecting them from one sedimentary unit [4].

Methodos of measurement

The orientation of a pebble is determined by the direction and angle of dip of its longest axis, and by the position of the maximum projection plane [5]. Various techniques used in the measurement work are based on the principle of determining the position of the pebble in the exposure in such a way as to permit the restoration of its position after taking it out of the sediment and on completion of the measurements required.

In the course of the studies here recorded, measurements concerning orientation were carried out after Karlstrom's [6] field technique. The equipment described by Karlstrom has been somewhat modified, making it possible to determine and restore the position of pebbles on a horizontal surface of gravels. Every pebble was measured for its azimuth and the angle of dip of its longest axis, as also for that of its intermediate axis. In different grades 100-pebble samples were studied, whereas in different shape classes, the samples studied were of 50 pebbles each.

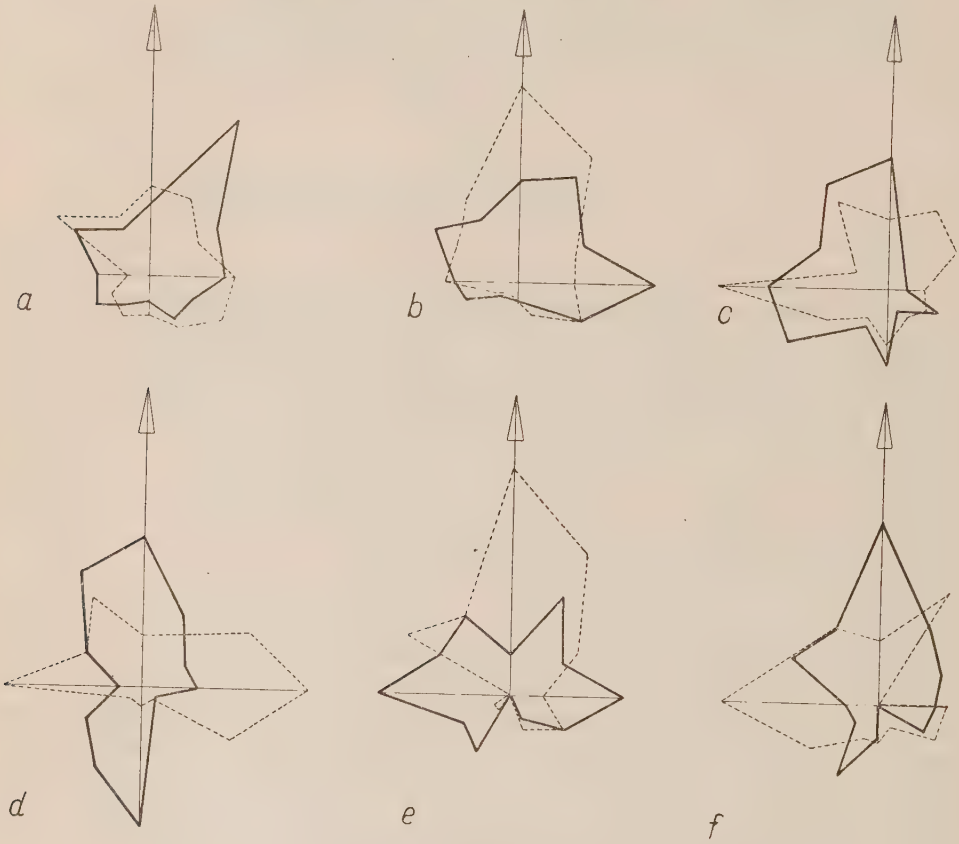

Fig. 1
Axis orientation of different grades of pebbles:
a — 128-256 mm. grade; *b* — 64-128 mm. grade; *c* — 32-64 mm. grade
Axis orientation of pebbles with different shapes:
d — prolate pebbles; *e* — ellipsoidal pebbles; *f* — discoidal pebbles;
Continuous line refers to longest axis orientation; broken line to intermediate axis orientation; arrows indicate direction of current.

The results of measurements were plotted by graphs in a polar coordinate pattern with a central projection. The points thus obtained were subsequently summed up in 30° sectors, with the direction of the current measured at the sampling site, taken as the centre of the first

sector. The number of points in the sectors is shown by graphs in star diagrams where the radial length corresponds to the number of points in the sector.

Preferred orientation of pebbles of different grades

Diagrams plotted for orientation measurements of various grades of pebbles show the following dependence of orientation on size:

Pebbles ranging in size from 128 to 256 mm. show the preferred direction of their longest axes to be in agreement with the direction of the current; the preferred direction of their intermediate axes is transversal to the direction of the current; the majority of the largest cross-section planes dip against the current.

Pebbles ranging in size from 64 to 128 mm. and making up the bulk of the sediment reveal a different orientation, i. e. the preferred direction of their longest axes is perpendicular to the direction of the current, while there occurs a markedly strong intermediate axes maximum disposed in the direction of the current. Virtually all the largest cross-section planes of pebbles dip upstream.

Pebbles ranging in size from 32 to 64 mm. do not show such distinct orientation in relation to the direction of the current. There occur several direction maxima of longest and intermediate axes. Connected with such is a lack of distinct orientation of the largest cross-section planes of pebbles. A steep dip of the axes of pebbles is encountered much more often in this grade of pebbles than in the two coarser grades.

Orientation of pebbles of various shapes

The orientation of pebbles is obviously dependent on their shape. Prolate pebbles show a very strong longest axes maximum, the axes being arranged to agree with the direction of the current. There is also an intermediate axes maximum, the axes here being arranged perpendicularly to the direction of the current. The longest axes may dip either up or down stream, in connection with which the largest cross-section planes of pebbles may also dip up or down stream. Ellipsoidal pebbles show the preferred longest axes direction to be perpendicular to the direction of the current, and have very strong intermediate axes maximum with axes arranged to agree with the direction of the current. Virtually all the largest cross-section planes of pebbles dip upstream.

Discoidal pebbles are not so strongly differentiated as regards their longest and intermediate axes orientation. This may be accounted for by the negligible difference in the axial length. The largest cross-section planes of pebbles here dip upstream.

Transportation and sedimentation of gravels and orientation of pebbles

Transportation of clastic material in the bed of the Dunajec is not continuous. The stream bed is covered with lag gravel which, at low-water, forms for the underlying sand and gravel stratum a protective shield against erosion. Transportation in the stream bed occurs only at times of very high water. The waters of the Dunajec rise with great violence. The gravel is set into motion on being struck by the flood. At times of maximal water rise, the gravel cover of the entire river bed is made to move, in accordance with the mechanism of river transport as described by Gilbert [7].

During transportation, the pebbles are made so to behave that their motion may involve the least possible loss of energy by the transporting agent. The predominant type of motion is the rolling of pebbles around their longest axes arranged horizontally and perpendicularly to the direction of the current [8]. As the water goes down and the velocity of the current decreases, the various gravel grades, beginning with the coarsest, are left behind the rolling mass of transported material. In that mass of finer material still carried forward, the largest pebbles, the transport of which is the first to cease, are stopped and poised down in a position of least resistance, i. e. with the smallest cross-section planes perpendicular to the direction of the movement. The longest axes are arranged parallel to the direction of the current. The slight deflection of the preferred longest axes direction from the direction of the current, as seen in the diagram plotted for the orientation data of the 128-256 mm. grade of pebbles, was caused probably by the close vicinity of the river bank. Kind and Kucharenko have experimentally shown the effect of this influence [9].

A further drop in velocity is accompanied by the deposition of the 64 to 128 mm. grade of pebbles, constituting the bulk of the sedimenting material. Here, the re-orientation of pebbles during their deposition is on a much smaller scale, and the orientation of the axes remains unaltered from that during the transportation. The smaller grades are deposited among larger pebbles and accommodate their position to the available free spaces. Thus the orientation of axes and of the largest cross-section planes of pebbles in relation to the direction of the current is not so very distinct in the smaller grades.

The influence of the shape of pebbles on their re-orientation is brought to bear at the moment of deposition. Prolate pebbles are subject to the most distinct re-orientation; moreover they also have a stable position even with their longest axes dipping downstream. Pebbles of other shapes are subject to weaker re-orientation, and retain their imbrication with their largest cross-section planes dipping upstream.

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